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Chain varieties of monoids

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Contents

Abstract

A variety of universal algebras is called a chain variety if its subvariety lattice is a chain. Non-group chain varieties of semigroups were completely classified by Sukhanov in 1982. Here we completely determine non-group chain varieties of monoids (referring to monoid varieties, we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the identity element). Even though the lattice of all monoid varieties embeds into the lattice of all semigroup varieties, surprisingly, the classification of non-group chain varieties in the monoid case turns out to be much more complicated than in the case of semigroups.

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1. Introduction and summary

There are many articles devoted to the examination of the lattice **SEM** of all semigroup varieties. An overview of this area is contained in the detailed survey [21]; see also the recent work [23] devoted to elements of **SEM** satisfying some special properties. In sharp contrast, the lattice **MON** of all monoid varieties has received much less attention over the years; when referring to monoid varieties, we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the identity element. As far as we know, there are only three papers containing substantial results on this subject. We have in mind the article [7] where the lattice of commutative monoid varieties is completely described, the article [24] which contains a complete description of the lattice of band monoid varieties, and the article [19] where an example of a monoid variety without covers in the lattice **MON** is found.

Recently, the situation has begun to change gradually. The papers [8, 9, 12–16] are mainly devoted to examination of identities of monoids but also contain some results about lattices of varieties. Moreover, [9] contains some results about the lattice **MON** that are of independent interest.

Thus nowadays, interest in the lattice **MON** has grown. Nevertheless, many questions in this area remain open. For example, it is known that the lattice **MON** is not modular (see, e.g., [12, Proposition 4.1] or Fig. 2.1b) below), but it was unknown up to the recent time whether this lattice satisfied some non-trivial identity. Only recently did the first author give a negative answer to this question [6]. In contrast, the fact that the lattice **SEM** does not satisfy any non-trivial lattice identity has been known since the early 1970's [3,4].

The problem of describing monoid varieties with modular or even distributive subvariety lattice seems to be quite difficult. As a first step in this direction, it seems natural to consider the extreme strengthening of the distributive law, namely the property of being a chain. Varieties whose subvariety lattice is a chain are called *chain varieties*. Non-group chain varieties of semigroups were listed by Sukhanov [22] (see Fig. 7.2 in Chapter 7 below), while locally finite chain group varieties were completely determined by Artamonov [2]. Note that the problem of completely describing arbitrary chain varieties of groups seems to be extremely difficult. This is confirmed by the fact that there are uncountably many periodic non-locally finite varieties of groups with 3-element subvariety lattice [11].

Some non-trivial examples of chain varieties of monoids appeared in [8, 12, 15]. However, chain monoid varieties have not been systematically studied so far. In this paper we obtain a complete description of non-group chain varieties of monoids. Note that it is verified in [8] that there exists a non-finitely based chain variety of monoids. This seems to be quite unexpected. For comparison, all non-group chain semigroup varieties and locally finite chain group varieties are finitely based. This follows from the results of [2, 22] mentioned above. Note also that by the result of [11] mentioned above, there exist non-finitely based non-locally finite chain varieties of groups. But explicit examples of such varieties have not yet been specified.

In order to formulate the main result of the article, we need some notation. We denote by F the free semigroup over a countably infinite alphabet A. Elements of both F and A are denoted by small Latin letters. However, elements of F for which it is not known exactly that they belong to A are written in bold. As usual, elements of F and of A are called *words* and *letters* respectively. The symbol F^1 stands for the semigroup F with a new identity element adjoined. We treat this identity element as the empty word and denote it by λ. We connect two sides of identities by the symbol $\approx$ and use $=$ for equality. We introduce notation for the following three identities:

$$\sigma_1 : \; xyzxty \approx yxzxty,$$
$$\sigma_2 : \; xtyzxy \approx xtyzyx,$$
$$\gamma_1 : \; y_1y_0x_1y_1x_0x_1 \approx y_1y_0y_1x_1x_0x_1.$$

Note that the identities σ_1 and σ_2 are dual to each other. The identity γ_1 belongs to a countably infinite series of identities γ_k that will be defined in Section 6.1. For an identity system Σ, we denote by $\operatorname{var}\Sigma$ the variety of monoids given by Σ. Let us fix notation for the following varieties:

$$\begin{aligned}
\mathbf{C}_n &= \operatorname{var}\{x^n \approx x^{n+1},\, xy \approx yx\} \quad \text{where } n \geq 2,\\
\mathbf{D} &= \operatorname{var}\{x^2 \approx x^3,\, x^2y \approx yx^2,\, \sigma_1,\, \sigma_2,\, \gamma_1\},\\
\mathbf{K} &= \operatorname{var}\{xyx \approx xyx^2,\, x^2y^2 \approx y^2x^2,\, x^2y \approx x^2yx\},\\
\mathbf{LRB} &= \operatorname{var}\{xy \approx xyx\},\\
\mathbf{N} &= \operatorname{var}\{x^2y \approx yx^2,\, x^2yz \approx xyxzx,\, \sigma_2,\, \gamma_1\},\\
\mathbf{RRB} &= \operatorname{var}\{yx \approx xyx\}.
\end{aligned}$$

To define one more variety, we need some additional notation. For every natural number n, we denote by S_n the full symmetric group on $\{1, \ldots, n\}$. For arbitrary permutations $\pi, \tau \in S_n$, we put

$$\mathbf{w}_n(\pi,\tau) = \Bigl(\prod_{i=1}^{n} z_it_i\Bigr) x \Bigl(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr) x \Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr),$$
$$\mathbf{w}_n'(\pi,\tau) = \Bigl(\prod_{i=1}^{n} z_it_i\Bigr) x^2 \Bigl(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr) \Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr).$$

Note that the words $\mathbf{w}_n(\pi,\tau)$ and $\mathbf{w}_n'(\pi,\tau)$ with the trivial permutations π and τ appeared earlier in [8, proof of Proposition 5.5]. Put

$$\mathbf{L} = \operatorname{var}\{x^2y \approx yx^2,\, xyxzx \approx x^2yz,\, \sigma_1,\, \sigma_2,\, \mathbf{w}_n(\pi,\tau) \approx \mathbf{w}_n'(\pi,\tau) \mid n \in \mathbb{N},\, \pi,\tau \in S_n\}.$$

If $\mathbf{X}$ is a monoid variety then we denote by $\overleftarrow{\mathbf{X}}$ the variety *dual to* $\mathbf{X}$, i.e. the variety consisting of all monoids antiisomorphic to monoids from $\mathbf{X}$.

The main result of the paper is

Theorem 1.1. *A non-group monoid variety is a chain variety if and only if it is contained in one of the varieties* $\mathbf{C}_n$ *for some* $n \geq 2$, $\mathbf{D}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$ *or* $\mathbf{RRB}$.

The complete list of all non-group chain varieties of monoids will be given in Corollary 7.1. The unique non-finitely based non-group chain variety of monoids mentioned above is $\mathbf{L}$ (see Corollary 4.8).

A minimal non-chain variety is called a *just-non-chain* variety. It is noted in [22, Corollary 2] that, among non-group varieties in $\mathbf{SEM}$, any chain variety is contained in some maximal chain variety and any non-chain variety contains some just-non-chain subvariety. However, similar results do not hold for non-group varieties in $\mathbf{MON}$. Specifically, the varieties $\mathbf{C}_3, \mathbf{C}_4, \ldots$ are not contained in any maximal chain variety (see Fig. 7.1 in Chapter 7), while it follows from Theorem 1.1 that there is a non-chain variety of monoids that does not contain any just-non-chain subvariety (see Corollary 7.4).

In [22] non-group chain varieties of semigroups were described in two ways. The first one is a description in the identity language. Theorem 1.1 is an analogue of this result in the case of monoids. The second way is by presenting the full list of non-group just-non-chain varieties of semigroups; this gives a characterization of chain varieties because, in view of [22, Corollary 2], a non-group variety of semigroups is a chain variety if and only if it does not contain any just-non-chain subvariety. As mentioned in the preceding paragraph, an analogous claim is false for monoids. Therefore, the second way of describing chain varieties is not applicable in the case of monoids. For this reason, we do not consider just-non-chain monoid varieties here.

The article consists of seven chapters. Chapter 2 contains definitions, notation and auxiliary results. In Chapter 3 we introduce new notions and notation and prove a number of results of technical character. These notions and results play a significant role in the proof of Theorem 1.1. Chapter 4 is devoted to the proof of the "only if" part of Theorem 1.1, while the "if" part is verified in Chapters 5 and 6. Finally, in Chapter 7 some corollaries of Theorem 1.1 and of its proof are established.

2. Preliminaries

A word is called a *semigroup word* if it does not contain the symbol of nullary operation 1. An identity is called a *semigroup identity* if both its sides are semigroup words. Note that an identity of the form $\mathbf{w} \approx 1$ is equivalent to the pair of identities $\mathbf{w}x \approx x\mathbf{w} \approx x$ where the letter x does not occur in $\mathbf{w}$. Further, any monoid satisfies the identities $\mathbf{u}\cdot 1 \approx 1\cdot \mathbf{u} \approx \mathbf{u}$ for any word $\mathbf{u}$. These observations allow us to assume that all identities that appear below are semigroup ones.

The *content* of a word $\mathbf{w}$, i.e., the set of all letters occurring in $\mathbf{w}$, is denoted by $\mathrm{con}(\mathbf{w})$. We denote by $\mathbf{SL}$ the variety of all semilattice monoids. The following statement is well known, but it has never appeared anywhere in this form, as far as we know. For the sake of completeness, we give its proof here.

Lemma 2.1. *For a monoid variety* $\mathbf{V}$*, the following are equivalent:*

(a) $\mathbf{V}$ *is a group variety;*
(b) $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\operatorname{con}(\mathbf{u}) \neq \operatorname{con}(\mathbf{v})$*;*
(c) $\mathbf{SL} \nsubseteq \mathbf{V}$.

Proof. The implication (a)⇒(c) is obvious.

The implication (c)⇒(b) follows immediately from the evident fact that the variety $\mathbf{SL}$ satisfies any identity $\mathbf{u} \approx \mathbf{v}$ with $\operatorname{con}(\mathbf{u}) = \operatorname{con}(\mathbf{v})$.

(b)⇒(a) By the hypothesis, there is a letter x that occurs in precisely one of the words $\mathbf{u}$ and $\mathbf{v}$. Let y be a letter with $y \notin \operatorname{con}(\mathbf{uv})$. Clearly, the identities $\mathbf{u}y \approx \mathbf{v}y$ and $y\mathbf{u} \approx y\mathbf{v}$ hold in $\mathbf{V}$. One can substitute 1 for all letters occurring in these identities except x and y. Then it follows that $\mathbf{V}$ satisfies $x^n y \approx y$ and $yx^n \approx y$ for some n. Hence $\mathbf{V}$ is a group variety. ∎

A letter is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if it occurs in $\mathbf{w}$ once [at least twice]. The set of all simple [multiple] letters in a word $\mathbf{w}$ is denoted by $\operatorname{sim}(\mathbf{w})$ [by $\operatorname{mul}(\mathbf{w})$]. The following statement is well known and can be easily verified.

Proposition 2.2. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{C}_2$ *if and only if*

$$\operatorname{sim}(\mathbf{u}) = \operatorname{sim}(\mathbf{v}) \quad \textit{and} \quad \operatorname{mul}(\mathbf{u}) = \operatorname{mul}(\mathbf{v}). \ \blacksquare \tag{2.1}$$

The following notion was introduced by Perkins [18] and has often appeared in the literature (see [8–10, 12, 15], for instance; in [9, Remark 2.4] there are a number of other references). Let W be a set of possibly empty words. We denote by $\overline{W}$ the set of all subwords of words from W and by $I(\overline{W})$ the set $F^1 \setminus \overline{W}$. It is clear that $I(\overline{W})$ is an ideal of F^1. Then $S(W)$ denotes the Rees quotient monoid $F^1/I(\overline{W})$. If $W = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ then we will write $S(\mathbf{w}_1, \dots, \mathbf{w}_k)$ rather than $S(\{\mathbf{w}_1, \dots, \mathbf{w}_k\})$.

A word $\mathbf{w}$ is called an *isoterm* for a class of semigroups if no semigroup in that class satisfies any non-trivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. The following statement is known in fact and plays an important role below.

Lemma 2.3. *Let* $\mathbf{V}$ *be a monoid variety and* W *a set of possibly empty words. Then* $S(W)$ *lies in* $\mathbf{V}$ *if and only if each word in* W *is an isoterm for* $\mathbf{V}$.

Proof. It is easy to verify that it suffices to consider the case when W consists of one word (see [8, paragraph after Lemma 3.3]). Then necessity is obvious, while sufficiency is proved in [10, Lemma 5.3]. ∎

The variety generated by a monoid M is denoted by $\operatorname{var} M$.

Lemma 2.4 ([1, Corollary 6.1.5]). $\mathbf{C}_{n+1} = \operatorname{var} S(x^n)$ *for any natural* n. ∎

Lemma 2.5. *Let* $\mathbf{V}$ *be a monoid variety and* n *a natural number. If* $\mathbf{C}_{n+1} \nsubseteq \mathbf{V}$ *then* $\mathbf{V}$ *satisfies an identity* $x^n \approx x^{n+m}$ *for some* m.

Proof. We can assume that $\mathbf{V}$ is not a group variety because the conclusion is evident otherwise. Lemmas 2.3 and 2.4 apply with the conclusion that the variety $\mathbf{V}$ satisfies a non-trivial identity of the form $x^n \approx \mathbf{w}$. Then $\operatorname{con}(\mathbf{w}) = \{x\}$ by Lemma 2.1, whence $\mathbf{w} = x^k$ for some $k \neq n$. Clearly, the identity $x^n \approx x^k$ implies $x^n \approx x^{n+m}$ for some m. Thus, the variety $\mathbf{V}$ satisfies the identity $x^n \approx x^{n+m}$. ∎

As in the case of semigroups, a variety of monoids is called *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups). It is well known that a variety is completely regular if and only if it satisfies an identity $x \approx x^{m+1}$ for some m. This observation, together with Lemma 2.5 and the evident fact that the variety $\mathbf{C}_2$ is not completely regular, implies

Corollary 2.6. *A monoid variety* $\mathbf{V}$ *is completely regular if and only if* $\mathbf{C}_2 \nsubseteq \mathbf{V}$. ∎

For any natural number k, we denote by $\mathbf{D}_k$ the subvariety of $\mathbf{D}$ given within $\mathbf{D}$ by the identity $x^2y_1y_2\cdots y_k \approx xy_1xy_2x\cdots xy_kx$. The proof of Proposition 4.1 in [15] implies

Lemma 2.7. $\mathbf{D}_1 = \operatorname{var} S(xy)$ *and* $\mathbf{D}_{n+1} = \operatorname{var} S(xy_1xy_2x\cdots xy_nx)$ *for any natural* n. ∎

We denote by $\mathbf{T}$ the trivial variety of monoids. The subvariety lattice of a monoid variety $\mathbf{X}$ is denoted by $L(\mathbf{X})$. Proposition 4.1 of [15] and its proof readily imply

Lemma 2.8. *The lattice* $L(\mathbf{D})$ *is the chain*

$$\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots \subset \mathbf{D}. \ ∎$$

The following statement follows immediately from [24, Proposition 4.7].

Lemma 2.9.

(i) *The lattice* $L(\mathbf{LRB} \vee \mathbf{RRB})$ *has the form shown in Fig.* 2.1(a).
(ii) *Every variety of band monoids either contains the variety* $\mathbf{LRB} \vee \mathbf{RRB}$ *or is contained in this variety.* ∎

Put

$$\mathbf{E} = \operatorname{var}\{x^2 \approx x^3,\ x^2y \approx xyx,\ x^2y^2 \approx y^2x^2\}.$$

The following lemma is verified in [12, Proposition 4.1(i) and Lemma 3.3(iv)].

Lemma 2.10.

(i) *The lattice* $L(\mathbf{LRB} \vee \mathbf{C}_2)$ *has the form shown in Fig.* 2.1(b).
(ii) $\mathbf{LRB} \vee \mathbf{C}_2 = \operatorname{var}\{x^2 \approx x^3,\ x^2y \approx xyx\}$. ∎

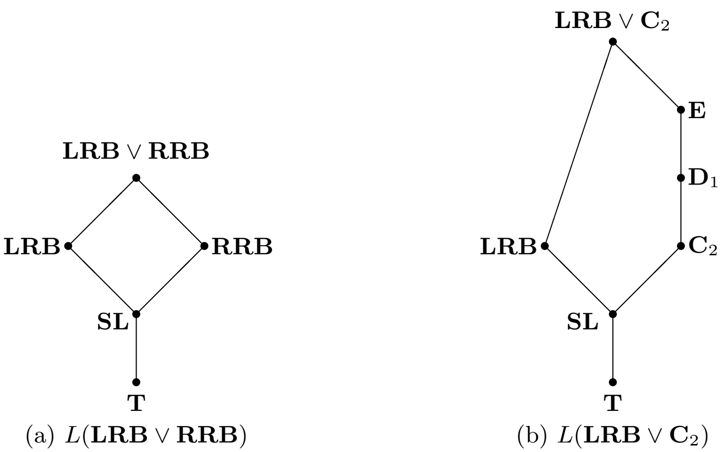

Fig. 2.1. The lattices $L(\mathbf{LRB} \vee \mathbf{RRB})$ and $L(\mathbf{LRB} \vee \mathbf{C}_2)$

Let $\mathbf{w}$ be a word and x a letter. We denote by $\operatorname{occ}_x(\mathbf{w})$ the number of occurrences of x in $\mathbf{w}$. If $x \in \operatorname{con}(\mathbf{w})$ and $i \le \operatorname{occ}_x(\mathbf{w})$ then $\ell_i(\mathbf{w}, x)$ denotes the length of the minimal prefix $\mathbf{p}$ of $\mathbf{w}$ with $\operatorname{occ}_x(\mathbf{p}) = i$.

Example 2.11. If $\mathbf{w} = xyx^2zy$ then, evidently, $\operatorname{occ}_x(\mathbf{w}) = 3$, $\operatorname{occ}_y(\mathbf{w}) = 2$ and $\operatorname{occ}_z(\mathbf{w}) = 1$. Further, the shortest prefixes $\mathbf{p}$ of $\mathbf{w}$ with $\operatorname{occ}_x(\mathbf{p}) = 1$, $\operatorname{occ}_x(\mathbf{p}) = 2$ and $\operatorname{occ}_x(\mathbf{p}) = 3$ are x, xyx and xyx^2 respectively, whence $\ell_1(\mathbf{w}, x) = 1$, $\ell_2(\mathbf{w}, x) = 3$ and $\ell_3(\mathbf{w}, x) = 4$. Analogously, $\ell_1(\mathbf{w}, y) = 2$, $\ell_2(\mathbf{w}, y) = 6$ and $\ell_1(\mathbf{w}, z) = 5$.

Below we often deal with inequalities like $\ell_i(\mathbf{w}, x) < \ell_j(\mathbf{w}, y)$. Clearly, this inequality means simply that the ith occurrence of x in $\mathbf{w}$ precedes the jth occurrence of y in $\mathbf{w}$.

If $\mathbf{w}$ is a word and X is a set of letters then $\mathbf{w}_X$ denotes the word obtained from $\mathbf{w}$ by deleting all letters from X. If $X = \{x\}$ then we write $\mathbf{w}_x$ rather than $\mathbf{w}_{\{x\}}$.

Lemma 2.12. *If a non-commutative variety of monoids* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *such that the claim* (2.1) *holds then*

$$\mathbf{u}_{\operatorname{mul}(\mathbf{u})} = \mathbf{v}_{\operatorname{mul}(\mathbf{u})}. \tag{2.2}$$

Proof. According to (2.1), $\operatorname{sim}(\mathbf{u}) = \operatorname{sim}(\mathbf{v})$ and $\operatorname{mul}(\mathbf{u}) = \operatorname{mul}(\mathbf{v})$. It is evident that (2.2) holds whenever $\operatorname{sim}(\mathbf{u})$ contains < 2 letters. Suppose now that $\operatorname{sim}(\mathbf{u})$ contains at least two different letters and (2.2) is false. Then there are letters $x, y \in \operatorname{sim}(\mathbf{u})$ such that $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, y)$ and $\ell_1(\mathbf{v}, x) > \ell_1(\mathbf{v}, y)$. One can substitute 1 for all letters occurring in the identity $\mathbf{u} \approx \mathbf{v}$ except x and y. Then we obtain $xy \approx yx$, contradicting the fact that $\mathbf{V}$ is non-commutative. ∎

Proposition 2.13. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{D}_1$ *if and only if* (2.1) *and* (2.2) *are true.*

Proof. Necessity. The inclusion $\mathbf{C}_2 \subseteq \mathbf{D}_1$ and Proposition 2.2 imply that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies (2.1). Since the variety $\mathbf{D}_1$ is non-commutative, Lemma 2.12 implies that (2.2) holds too.

Sufficiency. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies (2.1) and (2.2). Let $\operatorname{sim}(\mathbf{u}) = \{y_1, \dots, y_m\}$. We may assume without loss of generality that

$$\mathbf{u} = \mathbf{u}_0 y_1 \mathbf{u}_1 y_2 \mathbf{u}_2 \cdots y_m \mathbf{u}_m$$

where $\operatorname{con}(\mathbf{u}_0 \mathbf{u}_1 \cdots \mathbf{u}_m) = \operatorname{mul}(\mathbf{u})$. It follows from (2.1) that $\operatorname{sim}(\mathbf{v}) = \{y_1, y_2, \dots, y_m\}$. Moreover, $\mathbf{v} = \mathbf{v}_0 y_1 \mathbf{v}_1 y_2 \mathbf{v}_2 \cdots y_m \mathbf{v}_m$ by (2.2). We can apply (2.1) again to conclude that $\operatorname{con}(\mathbf{u}_0 \mathbf{u}_1 \cdots \mathbf{u}_m) = \operatorname{con}(\mathbf{v}_0 \mathbf{v}_1 \cdots \mathbf{v}_m)$. Now it is easy to see that the identity system $\{x^2 \approx x^3, x^2 y \approx xyx \approx yx^2\}$ implies the identities

$$\mathbf{u} = \mathbf{u}_0 y_1 \mathbf{u}_1 y_2 \mathbf{u}_2 \cdots y_m \mathbf{u}_m \approx \mathbf{v}_0 y_1 \mathbf{v}_1 y_2 \mathbf{v}_2 \cdots y_m \mathbf{v}_m = \mathbf{v},$$

whence $\mathbf{D}_1$ satisfies $\mathbf{u} \approx \mathbf{v}$. ∎

Lemma 2.14. *If a variety of monoids* $\mathbf{V}$ *is non-completely regular and non-commutative then* $\mathbf{D}_1 \subseteq \mathbf{V}$.

Proof. Suppose that $\mathbf{D}_1 \nsubseteq \mathbf{V}$. Then there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but is false in $\mathbf{D}_1$. Corollary 2.6 implies that $\mathbf{C}_2 \subseteq \mathbf{V}$. Then $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{C}_2$, whence (2.1) holds by Proposition 2.2. Now Lemma 2.12 and the assumption that $\mathbf{V}$ is non-commutative imply (2.2). Hence Proposition 2.13 applies, and we conclude that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{D}_1$, a contradiction. ∎

Lemma 2.15. *If* $\mathbf{X}$ *is a non-completely regular variety of monoids and* $\mathbf{D}_{n+1} \nsubseteq \mathbf{X}$ *for some* n *then* $\mathbf{X}$ *satisfies an identity of the form*

$$xy_1xy_2x\cdots xy_nx \approx x^{k_1}y_1x^{k_2}y_2x^{k_2}\cdots x^{k_n}y_nx^{k_{n+1}} \tag{2.3}$$

where $k_i > 1$ *for some* i.

Proof. If $\mathbf{X}$ is commutative then it satisfies the identity

$$xy_1xy_2x\cdots xy_nx \approx x^{n+1}y_1y_2\cdots y_n,$$

and we are done. Suppose now that $\mathbf{X}$ is non-commutative. Then it satisfies a non-trivial identity of the form $xy_1xy_2x\cdots xy_nx \approx \mathbf{w}$ by Lemmas 2.3 and 2.7. Now Lemma 2.14 applies, showing that $\mathbf{D}_1 \subseteq \mathbf{X}$. According to Proposition 2.13,

$$\mathbf{w} = x^{k_1}y_1x^{k_2}y_2x^{k_2}\cdots y_nx^{k_{n+1}}.$$

If $k_i > 1$ for some i then we are done. Suppose that $k_i \le 1$ for all i. There is $1 \le i \le n+1$ with $k_i = 0$ because the identity $xy_1xy_2x\cdots xy_nx \approx \mathbf{w}$ is trivial otherwise. Substitute xy_i for y_i in this identity for all i such that $k_i = 0$. If $k_{n+1} = 0$ then we multiply the resulting identity by x on the right. Thus, we obtain an identity of the form (2.3) where $k_i > 1$ for some i. ∎

3. k-decomposition of a word and related notions

Here we introduce a series of notions and examine their properties. These notions and results play a key role in the most complicated part of the proof of Theorem 1.1 in Chapter 6.

For a word $\mathbf{u}$ and letters $x_1, \dots, x_k \in \mathrm{con}(\mathbf{u})$, let $\mathbf{u}(x_1, \dots, x_k)$ denote the word obtained from $\mathbf{u}$ by retaining the letters $x_1, \dots, x_k$. Equivalently,

$$\mathbf{u}(x_1, \dots, x_k) = \mathbf{u}_{\mathrm{con}(\mathbf{u}) \setminus \{x_1, \dots, x_k\}}.$$

Let $\mathbf{w}$ be a word and $\mathrm{sim}(\mathbf{w}) = \{t_1, \dots, t_m\}$. We can assume without loss of generality that $\mathbf{w}(t_1, \dots, t_m) = t_1 \cdots t_m$. Then

$$\mathbf{w} = t_0\mathbf{w}_0t_1\mathbf{w}_1\cdots t_m\mathbf{w}_m \tag{3.1}$$

where $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are possibly empty words and $t_0 = \lambda$. The words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are called 0-*blocks* of $\mathbf{w}$, while $t_0, t_1, \dots, t_m$ are said to be 0-*dividers* of $\mathbf{w}$. The representation of $\mathbf{w}$ as a product of alternating 0-dividers and 0-blocks, starting with the 0-divider t_0 and ending with the 0-block $\mathbf{w}_m$, is called the 0-*decomposition* of $\mathbf{w}$.

Let now k be a natural number. We define the k-decomposition of $\mathbf{w}$ by induction on k. Let (3.1) be the $(k-1)$-decomposition of $\mathbf{w}$ with $(k-1)$-blocks $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ and $(k-1)$-dividers $t_0, t_1, \dots, t_m$. For any $i = 0, 1, \dots, m$, let $s_{i1}, \dots, s_{ir_i}$ be all simple letters in $\mathbf{w}_i$ that do not occur in the word $\mathbf{w}$ to the left of $\mathbf{w}_i$. We can assume that $\mathbf{w}_i(s_{i1}, \dots, s_{ir_i}) = s_{i1} \cdots s_{ir_i}$. Then

$$\mathbf{w}_i = \mathbf{v}_{i0}s_{i1}\mathbf{v}_{i1}s_{i2}\mathbf{v}_{i2}\cdots s_{ir_i}\mathbf{v}_{ir_i} \tag{3.2}$$

for possibly empty words $\mathbf{v}_{i0}, \mathbf{v}_{i1}, \dots, \mathbf{v}_{ir_i}$. Put $s_{i0} = t_i$. The words $\mathbf{v}_{i0}, \mathbf{v}_{i1}, \dots, \mathbf{v}_{ir_i}$ are called *k-blocks* of the word $\mathbf{w}$, while the letters $s_{i0}, s_{i1}, \dots, s_{ir_i}$ are said to be *k-dividers* of $\mathbf{w}$.

Remark 3.1. Note that only the first occurrence of a letter in a given word might be a k-divider of this word for some k. In view of this observation, below we use expressions like "a letter x is (or is not) a k-divider of a word $\mathbf{w}$" meaning that the first occurrence of x in $\mathbf{w}$ has the specified property.

For any $i = 0, 1, \dots, m$, we represent the $(k-1)$-block $\mathbf{w}_i$ in the form (3.2). As a result, we obtain the representation of $\mathbf{w}$ as a product of alternating k-dividers and k-blocks, starting with the k-divider $s_{00} = t_0$ and ending with the k-block $\mathbf{v}_{mr_m}$. This representation is called the *k-decomposition* of $\mathbf{w}$.

Remark 3.2. Since the length of $\mathbf{w}$ is finite, there is a number k such that the k-decomposition of $\mathbf{w}$ coincides with its n-decompositions for all $n > k$.

For the reader's convenience, we illustrate the notions of k-blocks, k-dividers and k-decomposition of a word by

Example 3.3. Let $\mathbf{w} = xyxzytszxs$. The unique simple letter in $\mathbf{w}$ is t. Therefore, the 0-decomposition of $\mathbf{w}$ is

$$\lambda \cdot \underline{xyxzy} \cdot t \cdot \underline{szxs} \tag{3.3}$$

(throughout this example we underline blocks to distinguish them from dividers). The unique simple letter of the leftmost 0-block $xyxzy$ is z; the 0-block $szxs$ contains two simple letters, namely z and x, but both occur in $\mathbf{w}$ to the left of this block. Therefore, the 1-decomposition of $\mathbf{w}$ is

$$\lambda \cdot \underline{xyx} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs}.$$

Analogous arguments show that the 2-decomposition of $\mathbf{w}$ is

$$\lambda \cdot \underline{x} \cdot y \cdot \underline{x} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs},$$

and if $k \geq 3$ then the k-decomposition of $\mathbf{w}$ is

$$\lambda \cdot \underline{\lambda} \cdot x \cdot \underline{\lambda} \cdot y \cdot \underline{x} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs}.$$

For a given word $\mathbf{w}$, a letter $x \in \mathrm{con}(\mathbf{w})$, a natural number $i \leq \mathrm{occ}_x(\mathbf{w})$ and an integer $k \geq 0$, we denote by $h_i^k(\mathbf{w}, x)$ the rightmost k-divider of $\mathbf{w}$ that precedes the ith occurrence of x in $\mathbf{w}$. The (possibly empty) letter $h_i^k(\mathbf{w}, x)$ is called the *(i,k)-restrictor* of the letter x in $\mathbf{w}$. This notion is illustrated by

Example 3.4. Let $\mathbf{w}$ be as in Example 3.3. The 0-decomposition of $\mathbf{w}$ has the form (3.3). We see that the rightmost 0-divider of $\mathbf{w}$ that precedes the first two occurrences of x, the two occurrences of y, and the first occurrences of z and t is λ, while the rightmost 0-divider of $\mathbf{w}$ that precedes the third occurrence of x, the second occurrence of z and both occurrences of s is t. This means that $h_1^0(\mathbf{w}, x) = h_2^0(\mathbf{w}, x) = \lambda$, $h_3^0(\mathbf{w}, x) = t$, $h_1^0(\mathbf{w}, y) = h_2^0(\mathbf{w}, y) = \lambda$, $h_1^0(\mathbf{w}, z) = \lambda$, $h_2^0(\mathbf{w}, z) = t$, $h_1^0(\mathbf{w}, s) = h_2^0(\mathbf{w}, s) = t$ and $h_1^0(\mathbf{w}, t) = \lambda$. Analogously, making use of Example 3.3, it is easy to find all other restrictors of letters in $\mathbf{w}$. The results are presented in Table 3.1.

Table 3.1. Restrictors of letters in the word $xyxzytszxs$

a	k	i	$h_i^k(\mathbf{w},a)$	a	k	i	$h_i^k(\mathbf{w},a)$
x	0	1	λ	z	0	1	λ
		2	λ			2	t
		3	t		1	1	λ
	1	1	λ			2	t
		2	λ		2	1	y
		3	t			2	t
	2	1	λ		≥ 3	1	y
		2	y			2	t
		3	t	s	0	1	t
	≥ 3	1	λ			2	t
		2	y		1	1	t
		3	t			2	t
y	0	1	λ		2	1	t
		2	λ			2	t
	1	1	λ		≥ 3	1	t
		2	z			2	t
	2	1	λ	t	0	1	λ
		2	z		1	1	z
	≥ 3	1	x		2	1	z
		2	z		≥ 3	1	z

LEMMA 3.5. *Let* $\mathbf{w}$ *be a word,* t *be a letter and* k, r *be numbers with* $r < k$.

(i) *If* t *is an* r*-divider of* $\mathbf{w}$ *then* t *is also a* k*-divider of* $\mathbf{w}$.
(ii) *If* $h_1^k(\mathbf{w},x) = h_2^k(\mathbf{w},x)$ *then* $h_1^r(\mathbf{w},x) = h_2^r(\mathbf{w},x)$ *as well.*
(iii) *If* $t_0\mathbf{w}_0t_1\mathbf{w}_1\cdots t_m\mathbf{w}_m$ *is the* k*-decomposition of* $\mathbf{w}$ *and* $m > 0$ *then* $t_m \in \mathrm{sim}(\mathbf{w})$.

Proof. Claims (i) and (ii) are obvious. To verify (iii), suppose that $t_m \in \mathrm{mul}(\mathbf{w})$. Then t_m is not a 0-divider of $\mathbf{w}$. Let p be the least natural number such that t_m is a p-divider but not a $(p-1)$-divider of $\mathbf{w}$. Evidently, $p \leq k$.

Suppose that $h_1^{p-1}(\mathbf{w},t_m) = h_2^{p-1}(\mathbf{w},t_m)$. This means that there are no $(p-1)$-dividers in $\mathbf{w}$ between the first and the second occurrences of t_m in $\mathbf{w}$. In other words, both these occurrences lie in the same $(p-1)$-block of $\mathbf{w}$. Therefore, t_m is not simple in this $(p-1)$-block. In particular, t_m is not a p-divider of $\mathbf{w}$, contradicting the choice of t_m. Thus, $h_1^{p-1}(\mathbf{w},t_m) \neq h_2^{p-1}(\mathbf{w},t_m)$. Note that the arguments of this paragraph are very typical. Below we use arguments like this many times, without repeating them explicitly.

Note that $t_m \neq h_2^{p-1}(\mathbf{w},t_m)$ because t_m is not a $(p-1)$-divider of $\mathbf{w}$. Put $t_{m+1} = h_2^{p-1}(\mathbf{w},t_m)$. Since $p-1 < k$, claim (i) implies that t_{m+1} is a k-divider of $\mathbf{w}$. The last k-divider of $\mathbf{w}$ is t_m. Therefore, the first occurrence of t_{m+1} in $\mathbf{w}$ precedes the first occurrence of t_m in $\mathbf{w}$. Therefore, $h_1^{p-1}(\mathbf{w},t_m) = t_{m+1} = h_2^{p-1}(\mathbf{w},t_m)$, a contradiction. ∎

For a given word $\mathbf{w}$ and a letter $x \in \mathrm{con}(\mathbf{w})$, we define a number called the *depth* of x in $\mathbf{w}$ and denoted by $D(\mathbf{w},x)$. If $x \in \mathrm{sim}(\mathbf{w})$ then we put $D(\mathbf{w},x) = 0$. Suppose now that $x \in \mathrm{mul}(\mathbf{w})$. If there is a natural k such that the first and the second occurrences of x in $\mathbf{w}$ lie in different $(k-1)$-blocks of $\mathbf{w}$ then the depth of x in $\mathbf{w}$ equals the minimal k with this property. Finally, if, for any natural k, the first and the second occurrences of x in $\mathbf{w}$ lie in the same k-block of $\mathbf{w}$ then we put $D(\mathbf{w},x)=\infty$. In other words, $D(\mathbf{w},x)=k$ if and only if $h_1^{k-1}(\mathbf{w},x)\ne h_2^{k-1}(\mathbf{w},x)$ and k is the least number with this property, while $D(\mathbf{w},x)=\infty$ if and only if $h_1^{k-1}(\mathbf{w},x)=h_2^{k-1}(\mathbf{w},x)$ for any k. This definition is illustrated by

Example 3.6. As in Examples 3.3 and 3.4, put $\mathbf{w}=xyxzzytszxs$. Here we systematically use information about restrictors of letters in $\mathbf{w}$ indicated in Table 3.1. In particular, in view of the table, $h_1^k(\mathbf{w},x)=\lambda$ for all k, while $h_2^0(\mathbf{w},x)=h_2^1(\mathbf{w},x)=\lambda$ and $h_2^2(\mathbf{w},x)=y$. Therefore, $D(\mathbf{w},x)=3$. Further, $h_1^0(\mathbf{w},y)=h_2^0(\mathbf{w},y)=\lambda$, $h_1^1(\mathbf{w},y)=\lambda$ and $h_2^1(\mathbf{w},y)=z$. Hence $D(\mathbf{w},y)=2$. The equalities $h_1^0(\mathbf{w},z)=\lambda$ and $h_2^0(\mathbf{w},z)=t$ imply that $D(\mathbf{w},z)=1$. Further, $h_1^k(\mathbf{w},s)=h_2^k(\mathbf{w},s)=t$ for each $k\ge 0$, whence $D(\mathbf{w},s)=\infty$. Finally, $D(\mathbf{w},t)=0$ because $t\in\mathrm{sim}(\mathbf{w})$.

The following criterion for a letter of a word to be a k-divider is often used in the proof of Theorem 1.1.

Lemma 3.7. *A letter t is a k-divider of a word $\mathbf{w}$ if and only if $D(\mathbf{w},t)\le k$.*

Proof. The statement is evident when $k=0$ because both the property of t being a 0-divider of $\mathbf{w}$ and the equality $D(\mathbf{w},t)=0$ are equivalent to t being simple in $\mathbf{w}$. Further, if $k>0$ then a letter t is a k-divider of $\mathbf{w}$ if and only if the first and the second occurrences of t lie in different $(k-1)$-blocks of $\mathbf{w}$. In turn, the last condition is equivalent to $h_1^{k-1}(\mathbf{w},t)\ne h_2^{k-1}(\mathbf{w},t)$, i.e., to $D(\mathbf{w},t)\le k$. ∎

Words $\mathbf{u}$ and $\mathbf{v}$ are said to be *k-equivalent* if they have the same set of k-dividers and these k-dividers appear in $\mathbf{u}$ and in $\mathbf{v}$ in the same order.

Lemma 3.8. *Let k be a non-negative integer. Words $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent if and only if* (2.1) *holds and, for any $x\in\mathrm{con}(\mathbf{uv})$, $h_1^k(\mathbf{u},x)=h_1^k(\mathbf{v},x)$ whenever either $D(\mathbf{u},x)\le k$ or $D(\mathbf{v},x)\le k$.*

Proof. Sufficiency. Suppose that

$$t_0\mathbf{u}_0t_1\mathbf{u}_1\cdots t_m\mathbf{u}_m \tag{3.4}$$

and $s_0\mathbf{v}_0s_1\mathbf{v}_1\cdots s_r\mathbf{v}_r$ are the k-decompositions of $\mathbf{u}$ and $\mathbf{v}$, respectively. Evidently, $t_0=s_0=\lambda$. If $m=r=0$ then the statement is evident. Let now $m>0$. In view of Lemma 3.7, $D(\mathbf{u},t_i)\le k$ for any $1\le i\le m$. By the hypothesis, this implies that $t_{i-1}=h_1^k(\mathbf{u},t_i)=h_1^k(\mathbf{v},t_i)$ for any $1\le i\le m$, whence t_{i-1} is a k-divider of $\mathbf{v}$. According to Lemma 3.5(iii), $t_m\in\mathrm{sim}(\mathbf{u})$. Then (2.1) implies that $t_m\in\mathrm{sim}(\mathbf{v})$, whence t_m is a 0-divider of $\mathbf{v}$. Now Lemma 3.5(i) applies to show that t_m is a k-divider of $\mathbf{v}$. So, the letters $t_1,\dots,t_m$ are k-dividers of $\mathbf{v}$, whence $m\le r$. By symmetry, $r\le m$. Thus $m=r$. Further, t_1 coincides with s_p for some p. If $p\ne 1$ then $h_1^k(\mathbf{v},t_1)\ne t_0$, contrary to $h_1^k(\mathbf{v},t_1)=h_1^k(\mathbf{u},t_1)=t_0$. So, $p=1$, and therefore $t_1=s_1$. By induction, we can verify that $t_j=s_j$ for any $j\le m$.

Necessity. Suppose that (3.4) is the k-decomposition of $\mathbf{u}$. Then the k-decomposition of $\mathbf{v}$ has the form

$$t_0\mathbf{v}_0t_1\mathbf{v}_1\cdots t_m\mathbf{v}_m. \tag{3.5}$$

Let $x \in \operatorname{con}(\mathbf{u})$ and $D(\mathbf{u}, x) \leq k$. Lemma 3.7 implies that $x = t_i$ for some $1 \leq i \leq m$. Therefore, $h_1^k(\mathbf{v}, x) = h_1^k(\mathbf{u}, x) = t_{i-1}$. Analogously, we verify that if $x \in \operatorname{con}(\mathbf{v})$ and $D(\mathbf{v}, x) \leq k$ then $h_1^k(\mathbf{v}, x) = h_1^k(\mathbf{u}, x)$. ■

Lemma 3.9. *Let* $\mathbf{w}$ *be a word,* x *be a letter multiple in* $\mathbf{w}$ *with* $D(\mathbf{w}, x) = k$ *and* t *be a* $(k-1)$*-divider of* $\mathbf{w}$.

(i) *If* $t = h_2^{k-1}(\mathbf{w}, x)$ *then* $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t)$.
(ii) *If* $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t) < \ell_2(\mathbf{w}, x)$ *then* $D(\mathbf{w}, t) = k - 1$*; if moreover* $k > 1$ *then* $\ell_2(\mathbf{w}, x) < \ell_2(\mathbf{w}, t)$.

Proof. (i) Suppose that $\ell_1(\mathbf{w}, t) < \ell_1(\mathbf{w}, x)$. Then the equality $t = h_2^{k-1}(\mathbf{w}, x)$ implies that $t = h_1^{k-1}(\mathbf{w}, x)$. Thus, $h_1^{k-1}(\mathbf{w}, x) = h_2^{k-1}(\mathbf{w}, x)$, which contradicts the assumption that $D(\mathbf{w}, x) = k$. So, $\ell_1(\mathbf{w}, x) \leq \ell_1(\mathbf{w}, t)$. Since t is a $(k-1)$-divider, Lemma 3.7 implies that $D(\mathbf{w}, t) \leq k - 1$. In particular, $D(\mathbf{w}, t) \neq D(\mathbf{w}, x)$, whence $t \neq x$. Therefore, $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t)$.

(ii) Suppose now that $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t) < \ell_2(\mathbf{w}, x)$. Put $r = D(\mathbf{w}, t)$. By Lemma 3.7, $r \leq k - 1$. If $D(\mathbf{w}, t) = r < k - 1$ then t is an r-divider by Lemma 3.7. Therefore, $t = h_2^r(\mathbf{w}, x)$. Further, $t \neq h_1^r(\mathbf{w}, x)$ because $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t)$. Thus, $h_1^r(\mathbf{w}, x) \neq h_2^r(\mathbf{w}, x)$. This means that $D(\mathbf{w}, x) \leq r + 1 < k$, a contradiction. So, $D(\mathbf{w}, t) = k - 1$.

Let now $k > 1$. Then $t \in \operatorname{mul}(\mathbf{w})$. Suppose that $\ell_2(\mathbf{w}, t) < \ell_2(\mathbf{w}, x)$. Put $s = h_2^{k-2}(\mathbf{w}, t)$. In view of (i), $\ell_1(\mathbf{w}, t) < \ell_1(\mathbf{w}, s)$. Arguments similar to those from the previous paragraph imply that $D(\mathbf{w}, s) = k - 2$. According to Lemma 3.7, s is a $(k-2)$-divider of $\mathbf{w}$. The choice of s guarantees that the first occurrence of s in $\mathbf{w}$ precedes the second occurrence of t. On the other hand, the second occurrence of t precedes the second occurrence of x. Thus, the first occurrence of s precedes the second occurrence of x. At the same time, the first occurrence of x precedes the first occurrence of s because $\ell_1(\mathbf{w}, x) < \ell_1(\mathbf{w}, t) < \ell_1(\mathbf{w}, s)$. Therefore, the first and second occurrences of x in $\mathbf{w}$ lie in different $(k-2)$-blocks. Hence, $D(\mathbf{w}, x) \leq k - 1$, a contradiction. ■

Lemma 3.10. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be words and* ℓ *be a natural number. Suppose that* (2.1) *holds and*

$$h_i^{\ell-1}(\mathbf{u}, x) = h_i^{\ell-1}(\mathbf{v}, x) \quad \text{for } i = 1, 2 \text{ and all } x \in \operatorname{con}(\mathbf{u}). \tag{3.6}$$

Then $\mathbf{u}$ *and* $\mathbf{v}$ *have the same set of* ℓ*-dividers.*

Proof. Let t be an arbitrary ℓ-divider of $\mathbf{u}$. If $t \in \operatorname{sim}(\mathbf{u})$ then $t \in \operatorname{sim}(\mathbf{v})$ by (2.1). Therefore, t is a 0-divider of $\mathbf{v}$. According to Lemma 3.5(i), t is an ℓ-divider of $\mathbf{v}$. Suppose now that $t \in \operatorname{mul}(\mathbf{u})$. Then (2.1) implies that $t \in \operatorname{mul}(\mathbf{v})$. Since t is an ℓ-divider of $\mathbf{u}$, $h_1^{\ell-1}(\mathbf{u}, t) \neq h_2^{\ell-1}(\mathbf{u}, t)$. Then $h_1^{\ell-1}(\mathbf{v}, t) \neq h_2^{\ell-1}(\mathbf{v}, t)$ by (3.6). This implies that t is an ℓ-divider of $\mathbf{v}$. Similarly we prove that if s is an ℓ-divider of $\mathbf{v}$ then s is an ℓ-divider of $\mathbf{u}$. ■

Lemma 3.11. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be words and* k *be a natural number. Suppose that* (2.1) *and* (3.6) *with* $\ell = k$ *hold. Then* (3.6) *holds with* $\ell = s$ *for any* $1 \leq s \leq k$.

Proof. If $k = 1$ then the assertion is valid by the hypothesis. Suppose now that $k > 1$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. In view of Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Let $s < k$ be least such that (3.6) with $\ell = s$ is false.

Then there exists a letter x such that $h_i^{s-1}(\mathbf{u},x) \ne h_i^{s-1}(\mathbf{v},x)$ for some $i \in \{1,2\}$. By the definition of $(i,s-1)$-restrictors, $h_i^{s-1}(\mathbf{u},x)$ and $h_i^{s-1}(\mathbf{v},x)$ are some $(s-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$ respectively. Lemma 3.5(i) implies that $(s-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$ are $(k-1)$-dividers of these words. Therefore, $h_i^{s-1}(\mathbf{u},x) = t_p$ and $h_i^{s-1}(\mathbf{v},x) = t_q$ for some $p \ne q$. We may assume without loss of generality that $p < q$. By the hypothesis, $h_i^{k-1}(\mathbf{u},x) = h_i^{k-1}(\mathbf{v},x)$, whence this $(i,k-1)$-restrictor of x coincide with t_n for some n. Clearly, $n \ge q$ because $s < k$. Since t_n precedes the ith occurrence of x in $\mathbf{u}$, we have $\ell_1(\mathbf{u},t_q) < \ell_i(\mathbf{u},x)$. Since t_p is an $(i,s-1)$-restrictor of x in $\mathbf{u}$, there are no $(s-1)$-dividers of $\mathbf{u}$ between the first occurrence of t_p and the ith occurrence of x in $\mathbf{u}$. In particular, t_q is not an $(s-1)$-divider of $\mathbf{u}$. Further, Lemma 3.7 implies that $D(\mathbf{u},t_q) > s-1$. In particular, $D(\mathbf{u},t_q) > 0$, whence $t_q \in \mathrm{mul}(\mathbf{u})$. If $s = 1$ then t_q is a 0-divider of $\mathbf{v}$, whence t_q is simple in $\mathbf{v}$. This contradicts (2.1). Thus, $s > 1$. This means that $h_1^{s-2}(\mathbf{u},t_q) = h_2^{s-2}(\mathbf{u},t_q)$. Since (3.6) holds with $\ell = s-1$, we obtain $h_1^{s-2}(\mathbf{v},t_q) = h_2^{s-2}(\mathbf{v},t_q)$. According to Lemma 3.5(ii), $h_1^{r-2}(\mathbf{v},t_q) = h_2^{r-2}(\mathbf{v},t_q)$ for all $r \le s$. Then $D(\mathbf{v},t_q) > s-1$. Lemma 3.7 implies that t_q is not an $(s-1)$-divider of $\mathbf{v}$, which contradicts $t_q = h_i^{s-1}(\mathbf{v},x)$. ∎

Lemma 3.12. *Let $\mathbf{u}$ and $\mathbf{v}$ be words and k be a natural number. Suppose that* (2.1) *and* (3.6) *with $\ell = k$ hold. Then, for any letter $x \in \mathrm{con}(\mathbf{u})$, $D(\mathbf{u},x) = k$ if and only if $D(\mathbf{v},x) = k$.*

Proof. In view of Lemma 3.11, (3.6) holds with $\ell = s$ for any $1 \le s \le k$. Suppose that $D(\mathbf{u},x) = k$. This implies that

$$h_1^{s-1}(\mathbf{v},x) = h_1^{s-1}(\mathbf{u},x) = h_2^{s-1}(\mathbf{u},x) = h_2^{s-1}(\mathbf{v},x)$$

whenever $1 \le s < k$ but

$$h_1^{k-1}(\mathbf{v},x) = h_1^{k-1}(\mathbf{u},x) \ne h_2^{k-1}(\mathbf{u},x) = h_2^{k-1}(\mathbf{v},x).$$

This implies that $D(\mathbf{v},x) = k$. By symmetry, if $D(\mathbf{v},x) = k$ then $D(\mathbf{u},x) = k$. ∎

Lemma 3.13. *Let $\mathbf{w}$ be a word, $r > 1$ be a number and y be a letter such that $D(\mathbf{w},y) = r-2$. Then if $\ell_1(\mathbf{w},z) < \ell_1(\mathbf{w},y)$ for some letter z with $D(\mathbf{w},z) \ge r$ then $\ell_2(\mathbf{w},z) < \ell_1(\mathbf{w},y)$.*

Proof. Let z be a letter with $\ell_1(\mathbf{w},z) < \ell_1(\mathbf{w},y)$ and $D(\mathbf{u},z) \ge r$. Lemma 3.7 implies that y is an $(r-2)$-divider of $\mathbf{w}$. Then if $\ell_1(\mathbf{u},y) < \ell_2(\mathbf{u},z)$ then the $(r-2)$-divider y is located between the first and the second occurrences of z in $\mathbf{u}$. This contradicts the equality $h_1^{r-2}(\mathbf{u},z) = h_2^{r-2}(\mathbf{u},z)$. The case $\ell_1(\mathbf{u},y) = \ell_2(\mathbf{u},z)$ is also impossible. Therefore, $\ell_2(\mathbf{w},z) < \ell_1(\mathbf{w},y)$. ∎

Below, in order to facilitate understanding of our considerations, we will sometimes write the number in brackets over a letter to indicate the number of occurrences of this letter in the given word; for instance, we may write

$$\mathbf{w} = \overset{(1)}{x_1}\,\overset{(1)}{x_2}\,\overset{(2)}{x_1}\,\overset{(1)}{x_3}\,\overset{(2)}{x_2}\,\overset{(3)}{x_1}\,.$$

Lemma 3.14. *Let $\mathbf{u} \approx \mathbf{v}$ be an identity and s be a natural number. Suppose that* (2.1) *and* (3.6) *with $\ell = s$ hold and there is a letter x_s such that $D(\mathbf{u},x_s) = s$. Then there exist letters $x_0, x_1, \dots, x_{s-1}$ such that $D(\mathbf{u},x_r) = D(\mathbf{v},x_r) = r$ for any $0 \le r < s$ and* *the identity* $\mathbf{u} \approx \mathbf{v}$ *has the form*

$$\begin{aligned}\mathbf{u}_{2s+1}\, \overset{(1)}{x_s}\, \mathbf{u}_{2s}\, \overset{(1)}{x_{s-1}}\, \mathbf{u}_{2s-1}\, \overset{(2)}{x_s}\, \mathbf{u}_{2s-2}\, \overset{(1)}{x_{s-2}}\, \mathbf{u}_{2s-3}\, \overset{(2)}{x_{s-1}}\, \mathbf{u}_{2s-4}\, \overset{(1)}{x_{s-3}} \\ \cdot \mathbf{u}_{2s-5}\, \overset{(2)}{x_{s-2}} \cdots \mathbf{u}_4\, \overset{(1)}{x_1}\, \mathbf{u}_3\, \overset{(2)}{x_2}\, \mathbf{u}_2\, \overset{(1)}{x_0}\, \mathbf{u}_1\, \overset{(2)}{x_1}\, \mathbf{u}_0 \\ \approx \mathbf{v}_{2s+1}\, \overset{(1)}{x_s}\, \mathbf{v}_{2s}\, \overset{(1)}{x_{s-1}}\, \mathbf{v}_{2s-1}\, \overset{(2)}{x_s}\, \mathbf{v}_{2s-2}\, \overset{(1)}{x_{s-2}}\, \mathbf{v}_{2s-3}\, \overset{(2)}{x_{s-1}}\, \mathbf{v}_{2s-4}\, \overset{(1)}{x_{s-3}} \\ \cdot \mathbf{v}_{2s-5}\, \overset{(2)}{x_{s-2}} \cdots \mathbf{v}_4\, \overset{(1)}{x_1}\, \mathbf{v}_3\, \overset{(2)}{x_2}\, \mathbf{v}_2\, \overset{(1)}{x_0}\, \mathbf{v}_1\, \overset{(2)}{x_1}\, \mathbf{v}_0\end{aligned} \tag{3.7}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2s+1}$ *and* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2s+1}$.

Proof. In view of Lemma 3.11, (3.6) holds with $\ell = r$ for any $1 \le r \le s$. We use this fact below without references.

Put $x_{s-1} = h_2^{s-1}(\mathbf{u}, x_s)$. Now (3.6) with $\ell = s$ implies that $h_2^{s-1}(\mathbf{v}, x_s) = h_2^{s-1}(\mathbf{u}, x_s) = x_{s-1}$. According to Lemma 3.9, $D(\mathbf{u}, x_{s-1}) = s-1$ and $\ell_j(\mathbf{u}, x_s) < \ell_j(\mathbf{u}, x_{s-1})$ for any $j = 1, 2$. Recall that $D(\mathbf{u}, x_s) = s$. According to Lemma 3.12, $D(\mathbf{v}, x_s) = s$. Now we apply Lemma 3.9 again to obtain $D(\mathbf{v}, x_{s-1}) = s-1$ and $\ell_j(\mathbf{v}, x_s) < \ell_j(\mathbf{v}, x_{s-1})$ for any $j = 1, 2$.

Further, put $x_{s-2} = h_2^{s-2}(\mathbf{u}, x_{s-1})$. According to Lemma 3.9, $D(\mathbf{u}, x_{s-2}) = s-2$ and $\ell_j(\mathbf{u}, x_{s-1}) < \ell_j(\mathbf{u}, x_{s-2})$ for any $j = 1, 2$. Now (3.6) with $\ell = s-1$ implies that $h_2^{s-2}(\mathbf{v}, x_{s-1}) = h_2^{s-2}(\mathbf{u}, x_{s-1}) = x_{s-2}$. We again apply Lemma 3.9 to obtain $D(\mathbf{v}, x_{s-2}) = s-2$ and $\ell_j(\mathbf{v}, x_{s-1}) < \ell_j(\mathbf{v}, x_{s-2})$ for any $j = 1, 2$. Since $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-1}) < \ell_1(\mathbf{u}, x_{s-2})$, we have $\ell_2(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-2})$ by Lemma 3.13. Analogously, $\ell_2(\mathbf{v}, x_s) < \ell_1(\mathbf{v}, x_{s-2})$.

Continuing, we define the letters $x_r = h_2^r(\mathbf{u}, x_{r+1})$ for $r = s-3, s-4, \ldots, 1$ and prove that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$, $\ell_j(\mathbf{u}, x_{r+1}) < \ell_j(\mathbf{u}, x_r)$, $\ell_j(\mathbf{v}, x_{r+1}) < \ell_j(\mathbf{v}, x_r)$ for any $j = 1, 2$, $\ell_2(\mathbf{u}, x_{r+2}) < \ell_1(\mathbf{u}, x_r)$ and $\ell_2(\mathbf{v}, x_{r+2}) < \ell_1(\mathbf{v}, x_r)$.

Finally, put $x_0 = h_2^0(\mathbf{u}, x_1)$. According to Lemma 3.9, $D(\mathbf{u}, x_0) = 0$ and $\ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$. Now (3.6) with $\ell = 1$ implies that $h_2^0(\mathbf{v}, x_1) = h_2^0(\mathbf{u}, x_1) = x_0$. We again apply Lemma 3.9 to obtain $D(\mathbf{v}, x_0) = 0$ and $\ell_1(\mathbf{v}, x_1) < \ell_1(\mathbf{v}, x_0)$. Since $\ell_1(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$, we have $\ell_2(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_0)$ by Lemma 3.13. Analogously, $\ell_2(\mathbf{v}, x_2) < \ell_1(\mathbf{v}, x_0)$.

In view of the above, the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2s+1}$. ■

LEMMA 3.15. *Let* $\mathbf{w} = y_1 \cdots y_n$ *where the letters* $y_1, \ldots, y_n$ *are not necessarily pairwise different. Further, let* $\mathbf{u} = \mathbf{u}'\xi(\mathbf{w})\mathbf{u}''$ *for some possibly empty words* $\mathbf{u}'$ *and* $\mathbf{u}''$ *and some endomorphism* ξ *of* F^1. *Put* $\xi(y_i) = \mathbf{w}_i$ *for all* $i = 1, \ldots, n$. *If* $D(\mathbf{w}, y_i) > 0$ *then the subword* $\mathbf{w}_i$ *of* $\mathbf{u}$ *contains no* r*-divider of* $\mathbf{u}$ *for any* $r < D(\mathbf{w}, y_i)$.

Proof. Let $1 \le i \le n$ and $D(\mathbf{w}, y_i) > 0$. Then $y_i \in \mathrm{mul}(\mathbf{w})$, whence $\mathrm{con}(\mathbf{w}_i) \subseteq \mathrm{mul}(\xi(\mathbf{w})) \subseteq \mathrm{mul}(\mathbf{u})$. This implies that $\mathbf{w}_i$ does not contain any 0-divider of $\mathbf{u}$. Let now $r > 0$ be least such that there exists i such that $D(\mathbf{w}, y_i) > r$ but $\mathbf{w}_i$ contains some r-divider t of $\mathbf{u}$. The choice of r and Lemma 3.7 imply that $D(\mathbf{u}, t) = r$. Clearly, $t \notin \mathrm{con}(\mathbf{w}_1 \cdots \mathbf{w}_{i-1})$, whence y_i differs from $y_1, \ldots, y_{i-1}$. Since $y_i \in \mathrm{mul}(\mathbf{w})$, there is some $j \ge i$ such that $\mathbf{w}_j$ contains the second occurrence of t in $\mathbf{u}$. Put $x = h_2^{r-1}(\mathbf{u}, t)$. In view of Lemma 3.9(i), $\ell_1(\mathbf{u}, t) < \ell_1(\mathbf{u}, x)$. Then there is $i \le \ell \le j$ such that $\mathbf{w}_\ell$ contains the $(r-1)$-divider x of $\mathbf{u}$. In view of the choice of r, $D(\mathbf{w}, y_\ell) \le r-1$. This implies that $y_i \neq y_\ell$, whence $\ell_1(\mathbf{w}, y_i) < \ell_1(\mathbf{w}, y_\ell)$. Further, since $y_i \in \mathrm{mul}(\mathbf{w})$, there is $p \geq j$ such that $y_i = y_p$. We note that $\ell < p$ because $y_p = y_i \neq y_\ell$. So, we obtain $\ell_1(\mathbf{w}, y_i) < \ell_1(\mathbf{w}, y_\ell) < \ell_2(\mathbf{w}, y_i)$. Lemma 3.7 implies that y_ℓ is an $(r-1)$-divider of $\mathbf{w}$, whence $h_1^{r-1}(\mathbf{w}, y_i) \neq h_2^{r-1}(\mathbf{w}, y_i)$, contrary to $D(\mathbf{w}, y_i) > r$. ■

4. The proof of the "only if" part

Throughout this chapter, $\mathbf{V}$ denotes a fixed non-group chain variety of monoids. We aim to verify that $\mathbf{V}$ is contained in one of the varieties listed in Theorem 1.1. The chapter is divided into three sections.

4.1. Reduction to the case when $\mathbf{D}_2 \subseteq \mathbf{V}$. A variety of monoids is called *aperiodic* if all its groups are singletons. Lemma 2.1 implies that $\mathbf{SL} \subseteq \mathbf{V}$. If $\mathbf{V}$ contains a non-trivial group then the variety generated by this group is incomparable with $\mathbf{SL}$. This contradicts $\mathbf{V}$ being is a chain variety. Therefore, $\mathbf{V}$ is aperiodic, whence it satisfies the identity $x^n \approx x^{n+1}$ for some n. If $\mathbf{V}$ is commutative then $\mathbf{V} \subseteq \mathbf{SL} \subseteq \mathbf{C}_2$ for $n = 1$ and $\mathbf{V} \subseteq \mathbf{C}_n$ otherwise.

Further, if $\mathbf{V}$ is a variety of band monoids then Lemma 2.9 and the observation that $\mathbf{V}$ cannot contain simultaneously the incomparable varieties $\mathbf{LRB}$ and $\mathbf{RRB}$ imply that $\mathbf{V}$ is contained in one of these two varieties.

Suppose now that $\mathbf{V}$ is non-commutative and is not a variety of band monoids. Then $\mathbf{V}$ is not completely regular because every aperiodic completely regular variety consists of bands. Then Lemma 2.14 implies that $\mathbf{D}_1 \subseteq \mathbf{V}$. To continue our considerations, we need several assertions.

LEMMA 4.1. *Let $\mathbf{X}$ be a monoid variety such that $\mathbf{D}_1 \subseteq \mathbf{X}$. Then either $\mathbf{X}$ satisfies an identity of the form*

$$x^s y x^t \approx y x^r \tag{4.1}$$

where $s \geq 1$, $t \geq 0$, $s + t \geq 2$ and $r \geq 2$, or, for any identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$, we have

$$h_1^0(\mathbf{u}, x) = h_1^0(\mathbf{v}, x) \quad \text{for all } x \in \mathrm{con}(\mathbf{u}). \tag{4.2}$$

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be an identity that holds in $\mathbf{X}$. The inclusion $\mathbf{D}_1 \subseteq \mathbf{X}$ and Proposition 2.13 imply (2.1) and (2.2). Hence if (3.4) is the 0-decomposition of $\mathbf{u}$ then the 0-decomposition of $\mathbf{v}$ has the form (3.5). Suppose that (4.2) is false. Then there is a letter $x \in \mathrm{mul}(\mathbf{u})$ such that $h_1^0(\mathbf{u}, x) \neq h_1^0(\mathbf{v}, x)$. Now (2.1) implies that $x \in \mathrm{mul}(\mathbf{v})$. Further, we may assume without loss of generality that there are $i < j$ such that $t_i = h_1^0(\mathbf{u}, x)$ and $t_j = h_1^0(\mathbf{v}, x)$. Substituting y for t_j and 1 for all letters occurring in the identity $\mathbf{u} \approx \mathbf{v}$ except x and t_j, we see that $\mathbf{X}$ satisfies an identity of the form (4.1) where $s \geq 1$, $t \geq 0$, $s + t \geq 2$ and $r \geq 2$. ■

When we make simultaneously several substitutions in some identity, say, substitute $\mathbf{u}_i$ for x_i for $i = 1, \dots, k$, then we will say for brevity that we perform the substitution

$$(x_1, \dots, x_k) \mapsto (\mathbf{u}_1, \dots, \mathbf{u}_k).4.2$$

Proposition 4.2. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{E}$ *if and only if* (2.1) *and* (4.2) *hold.*

Proof. Necessity. Suppose that $\mathbf{E}$ satisfies $\mathbf{u} \approx \mathbf{v}$. The inclusion $\mathbf{D}_1 \subseteq \mathbf{E}$ and Proposition 2.13 imply that this identity satisfies (2.1). Suppose that (4.2) is false. Then Lemma 4.1 shows that $\mathbf{E}$ satisfies an identity of the form (4.1) where $s \geq 1$, $t \geq 0$, $s + t \geq 2$ and $r \geq 2$. Consider the semigroup

$$P = \langle e, a \mid e^2 = e,\ ae = a,\ ea = 0\rangle = \{e, a, 0\}.$$

Note that $\mathbf{E}$ contains the monoid P^1, i.e., the semigroup P with a new identity element adjoined. Making the substitution $(x, y) \mapsto (e, a)$ in (4.1) results in the contradiction $0 = a$. Thus, P^1, and therefore $\mathbf{E}$, violates (4.1), a contradiction.

Sufficiency. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies (2.1) and (4.2). Let (3.4) be the 0-decomposition of $\mathbf{u}$. In view of Lemma 3.8, the 0-decomposition of $\mathbf{v}$ has the form (3.5). We are going to verify that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{E}$. Recall that $\mathbf{E}$ is given by the identity system

$$\{x^2 \approx x^3,\ x^2y \approx xyx,\ x^2y^2 \approx y^2x^2\}. \tag{4.3}$$

Put $X = \mathrm{con}(\mathbf{u}_0) = \{x_1, \dots, x_k\}$. Clearly, no block of any word $\mathbf{w}$ contains letters simple in $\mathbf{w}$. Therefore, we may assume without loss of generality that $\mathbf{u}_0 = x_1^2 \cdots x_k^2$.

We will use induction on the parameter m from (3.4) and (3.5).

Induction base. Let $m = 0$. Now (2.1) implies that $\mathrm{con}(\mathbf{u}_0) = \mathrm{con}(\mathbf{v}_0)$. Since the variety $\mathbf{E}$ satisfies the identity

$$x^2y^2 \approx y^2x^2, \tag{4.4}$$

it also satisfies $\mathbf{v}_0 \approx x_1^2 \cdots x_k^2$. Therefore, the identities

$$\mathbf{u} = t_0\mathbf{u}_0 = t_0x_1^2x_2^2 \cdots x_k^2 \approx t_0\mathbf{v}_0 = \mathbf{v}$$

hold in $\mathbf{E}$.

Induction step. Let now $m > 0$. The identity system (4.3) implies the identity

$$\mathbf{u} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1(\mathbf{u}_1)_X \cdots t_m(\mathbf{u}_m)_X.$$

By (4.2), $\mathrm{con}(\mathbf{u}_0) = \mathrm{con}(\mathbf{v}_0)$, whence (4.3) implies the identity

$$\mathbf{v} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1(\mathbf{v}_1)_X \cdots t_m(\mathbf{v}_m)_X.$$

Put $\mathbf{u}' = (\mathbf{u}_1)_X \cdots t_m(\mathbf{u}_m)_X$ and $\mathbf{v}' = (\mathbf{v}_1)_X \cdots t_m(\mathbf{v}_m)_X$. It is easy to verify that the identity $\mathbf{u}' \approx \mathbf{v}'$ satisfies (2.1) and (4.2). By the induction assumption, the identity $\mathbf{u}' \approx \mathbf{v}'$ holds in $\mathbf{E}$, whence this variety satisfies

$$\mathbf{u} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1\mathbf{u}' \approx t_0x_1^2x_2^2 \cdots x_k^2t_1\mathbf{v}' \approx \mathbf{v}.$$

Thus, $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{E}$. ■

Lemma 4.3. *Let* $\mathbf{X}$ *be a non-completely regular variety of monoids. If* $\mathbf{E} \nsubseteq \mathbf{X}$ *and* $\mathbf{X}$ *satisfies the identity*

$$x^2 \approx x^3 \tag{4.5}$$

then $\mathbf{X}$ *also satisfies the identity*

$$yx^2 \approx x^2yx^2. \tag{4.6}$$

Proof. If $\mathbf{X}$ is commutative then by (4.5), $\mathbf{X}$ satisfies the identities $yx^2 \approx yx^4 \approx x^2yx^2$. Suppose now that $\mathbf{X}$ is non-commutative. Then Lemma 2.14 implies that $\mathbf{D}_1 \subseteq \mathbf{X}$. Since $\mathbf{E} \nsubseteq \mathbf{X}$, there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but fails in $\mathbf{E}$. Then Proposition 4.2 shows that either (2.1) or (4.2) is false. Proposition 2.2 implies that (2.1) is true because $\mathbf{C}_2 \subseteq \mathbf{D}_1 \subseteq \mathbf{X}$. Therefore, (4.2) is false. Now Lemma 4.1 shows that $\mathbf{X}$ satisfies an identity of the form (4.1) where $s \ge 1$, $t \ge 0$, $s+t \ge 2$ and $r \ge 2$. Substitute x^2 for x in this identity. Since $\mathbf{X}$ satisfies (4.5), we conclude that (4.6) holds in $\mathbf{X}$. ∎

Let us return to the examination of a chain variety $\mathbf{V}$. Recall that we reduce considerations to the case when $\mathbf{D}_1 \subseteq \mathbf{V}$. Hence $\mathbf{C}_3 \nsubseteq \mathbf{V}$ because $\mathbf{C}_3$ and $\mathbf{D}_1$ are incomparable. Then Lemma 2.5 and the fact that $\mathbf{V}$ is aperiodic imply that the identity (4.5) holds in $\mathbf{V}$. Suppose now that $\mathbf{D}_2 \nsubseteq \mathbf{V}$. The variety $\mathbf{V}$ does not contain at least one of the incomparable varieties $\mathbf{E}$ and $\overleftarrow{\mathbf{E}}$. Assume without loss of generality that $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$. The dual of Lemma 4.3 then implies that $\mathbf{V}$ satisfies the identity

$$x^2y \approx x^2yx^2. \tag{4.7}$$

Further, Lemma 2.15 implies that the identity

$$xyx \approx x^qyx^r \tag{4.8}$$

with $q > 1$ or $r > 1$ holds in $\mathbf{V}$.

If $\mathbf{u}$ and $\mathbf{v}$ are words and ε is an identity then we will write $\mathbf{u} \stackrel{\varepsilon}{\approx} \mathbf{v}$ whenever the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε. If $q > 1$ then $\mathbf{V}$ satisfies the identities

$$xyx \stackrel{(4.8)}{\approx} x^qyx^r \stackrel{(4.7)}{\approx} x^qyx^{r+2} \stackrel{(4.5)}{\approx} x^2yx^2 \stackrel{(4.7)}{\approx} x^2y.$$

Recall that $\mathbf{V}$ satisfies (4.5) too. Then Lemma 2.10(ii) shows that $\mathbf{V} \subseteq \mathbf{LRB} \vee \mathbf{C}_2$. Since $\mathbf{V}$ is non-idempotent and chain, $\mathbf{V} \subseteq \mathbf{E}$ by Lemma 2.10(i). Therefore, $\mathbf{V} \subseteq \mathbf{K}$.

Suppose now that $q \le 1$. Then $r > 1$. If $q = 0$ then $\mathbf{V} \subseteq \mathbf{RRB} \vee \mathbf{C}_2$ by the dual of Lemma 2.10(ii) because $\mathbf{V}$ satisfies the identity (4.5). Since $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$ and $\mathbf{V}$ is not a variety of band monoids, it follows from the dual of Lemma 2.10(i) that $\mathbf{V} \subseteq \mathbf{D}_1 \subseteq \mathbf{D}$.

Let now $q = 1$. Then $\mathbf{V}$ satisfies the identity

$$xyx \approx xyx^2 \tag{4.9}$$

because it satisfies (4.5). Therefore, the identities $x^2yx \stackrel{(4.9)}{\approx} x^2yx^2 \stackrel{(4.7)}{\approx} x^2y$ hold in $\mathbf{V}$. Thus, $\mathbf{V}$ satisfies

$$x^2y \approx x^2yx. \tag{4.10}$$

Corollary 2.6 implies that $\mathbf{C}_2 \subseteq \mathbf{V}$. Therefore, $\mathbf{LRB} \nsubseteq \mathbf{V}$. Hence there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but fails in $\mathbf{LRB}$. The *initial part* of a word $\mathbf{w}$, denoted by $\mathrm{ini}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter. It is evident that an identity $\mathbf{a} \approx \mathbf{b}$ holds in $\mathbf{LRB}$ if and only if $\mathrm{ini}(\mathbf{a}) = \mathrm{ini}(\mathbf{b})$. Hence $\mathrm{ini}(\mathbf{u}) \ne \mathrm{ini}(\mathbf{v})$. Proposition 2.2 implies that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. Therefore, we can assume that there are letters $x, y \in \mathrm{con}(\mathbf{u})$ such that $\mathbf{u}(x, y) = x^sy\mathbf{w}_1$ and $\mathbf{v}(x, y) = y^tx\mathbf{w}_2$ where $s, t > 0$ and $\mathrm{con}(\mathbf{w}_1) = \mathrm{con}(\mathbf{w}_2) = \{x, y\}$. Let us substitute 1 for all letters except x and y in $\mathbf{u} \approx \mathbf{v}$. We find that $\mathbf{V}$ satisfies the identity $x^sy\mathbf{w}_1 \approx y^tx\mathbf{w}_2$. If $s = 1$ then we substitute x^2 for x in this identity and obtain an identity of the form $x^2y\mathbf{w}_1' \approx y^tx^2\mathbf{w}_2'$. Thus, we can assume that $s \ge 2$. Analogously, we can assume that $t \ge 2$. Moreover, the identity (4.5) allows us to assume that $s = t = 2$. Now we can apply (4.10) to deduce an identity of the form $x^2y^k \approx y^2x^m$ where $k, m > 1$. Moreover, (4.5) allows us to assume that $k = m = 2$. We thus find that (4.4) holds in $\mathbf{V}$. This means that $\mathbf{V} \subseteq \mathbf{K}$.

It remains to consider the case when $\mathbf{D}_2 \subseteq \mathbf{V}$.

4.2. Reduction to the case when $\mathbf{L} \subseteq \mathbf{V}$. Here we need some notation and a series of auxiliary assertions. Let n and m be non-negative integers such that $n + m > 0$. For any $\theta \in S_{n+m}$, we put

$$\mathbf{w}_{n,m}(\theta) = \Bigl(\prod_{i=1}^{n} z_i t_i\Bigr) x \Bigl(\prod_{i=1}^{n+m} z_{\theta(i)}\Bigr) x \Bigl(\prod_{i=n+1}^{n+m} t_i z_i\Bigr),$$

$$\mathbf{w}'_{n,m}(\theta) = \Bigl(\prod_{i=1}^{n} z_i t_i\Bigr) x^2 \Bigl(\prod_{i=1}^{n+m} z_{\theta(i)}\Bigr) \Bigl(\prod_{i=n+1}^{n+m} t_i z_i\Bigr).$$

Note that the words $\mathbf{w}_n(\pi, \tau)$ and $\mathbf{w}'_n(\pi, \tau)$ introduced in Chapter 1 are of the form $\mathbf{w}_{n,n}(\theta)$ and $\mathbf{w}'_{n,n}(\theta)$ respectively for an appropriate permutation $\theta \in S_{2n}$.

Lemma 4.4. *The variety $\mathbf{L}$ satisfies the identities*

$$\mathbf{w}_{n,m}(\theta) \approx \mathbf{w}'_{n,m}(\theta) \tag{4.11}$$

for all n, m and $\theta \in S_{n+m}$.

Proof. It suffices to verify that each identity of the form (4.11) follows from some identity of the form

$$\mathbf{w}_n(\pi, \tau) \approx \mathbf{w}'_n(\pi, \tau). \tag{4.12}$$

To do this, we fix an identity of the form (4.11), namely

$$\mathbf{p}_0 x \mathbf{q}_0 x \mathbf{r}_0 \approx \mathbf{p}_0 x^2 \mathbf{q}_0 \mathbf{r}_0$$

where $\mathbf{p}_0 = z_1 t_1 \cdots z_n t_n$, $\mathbf{q}_0 = z_{\theta(1)} \cdots z_{\theta(n+m)}$ and $\mathbf{r}_0 = t_{n+1} z_{n+1} \cdots t_{n+m} z_{n+m}$. The word $\mathbf{q}_0$ may be uniquely represented as

$$\mathbf{q}_0 = \mathbf{u}_1 \mathbf{v}_1 \cdots \mathbf{u}_k \mathbf{v}_k$$

where $\mathrm{con}(\mathbf{u}_1 \cdots \mathbf{u}_k) = \{z_1, \dots, z_n\}$ and $\mathrm{con}(\mathbf{v}_1 \cdots \mathbf{v}_k) = \{z_{n+1}, \dots, z_{n+m}\}$ (we mean here that $\mathbf{u}_1 = \lambda$ whenever $\theta(1) > n$, and $\mathbf{v}_k = \lambda$ whenever $\theta(n+m) \le n$). Each of the words $\mathbf{u}_1, \dots, \mathbf{u}_k$ (except $\mathbf{u}_1$ whenever $\mathbf{u}_1 = \lambda$) has the form $z_{j_1} \cdots z_{j_s}$ where $j_1, \dots, j_s \le n$, while each of the words $\mathbf{v}_1, \dots, \mathbf{v}_k$ (except $\mathbf{v}_k$ whenever $\mathbf{v}_k = \lambda$) has the form $z_{j_1} \cdots z_{j_s}$ where $j_1, \dots, j_s > n$.

Suppose first that $\mathbf{u}_1 = \lambda$. Let z and t be letters that do not occur in $\mathbf{p}_0\mathbf{q}_0\mathbf{r}_0 x$. Put $\mathbf{p}' = zt\mathbf{p}_0$, $\mathbf{q}' = z\mathbf{q}_0$ and $\mathbf{r}' = \mathbf{r}_0$. The identity $\mathbf{p}' x \mathbf{q}' x \mathbf{r}' \approx \mathbf{p}' x^2 \mathbf{q}' \mathbf{r}'$ evidently implies (4.11). Up to the evident renaming of letters, the identity $\mathbf{p}' x \mathbf{q}' x \mathbf{r}' \approx \mathbf{p}' x^2 \mathbf{q}' \mathbf{r}'$ has the form indicated in the previous paragraph with $\mathbf{u}_1 \ne \lambda$. Thus, we can assume that $\mathbf{u}_1 \ne \lambda$. Analogous arguments allow us to suppose that $\mathbf{v}_k \ne \lambda$.

Let now $\mathbf{u}_1 = z_{j_1} \cdots z_{j_s}$ with $j_1, \dots, j_s \le n$. Let z'_{j_1}, t'_{j_1}, $\dots$, $z'_{j_{s-1}}$, $t'_{j_{s-1}}$ be letters that do not occur in $\mathbf{p}_0\mathbf{q}_0\mathbf{r}_0 x$. Put $\mathbf{p}_1 = \mathbf{p}_0$. Denote by $\mathbf{q}_1$ the word obtained from $\mathbf{q}_0$ by replacing $\mathbf{u}_1$ with $z_{j_1} z'_{j_1} \cdots z_{j_{s-1}} z'_{j_{s-1}} z_{j_s}$. Finally, we put $\mathbf{r}_1 = \mathbf{r}_0 t'_{j_1} z'_{j_1} \cdots t'_{j_{s-1}} z'_{j_{s-1}}$. The identity $\mathbf{p}_0x\mathbf{q}_0x\mathbf{r}_0 \approx \mathbf{p}_0x^2\mathbf{q}_0\mathbf{r}_0$ follows from $\mathbf{p}_1x\mathbf{q}_1x\mathbf{r}_1 \approx \mathbf{p}_1x^2\mathbf{q}_1\mathbf{r}_1$ by substitution of 1 for $z'_{j_1}, t'_{j_1}, \dots, z'_{j_{s-1}}, t'_{j_{s-1}}$.

Further, let $\mathbf{v}_1 = z_{j_1} \cdots z_{j_s}$ where $j_1, \dots, j_s > n$. Let $z'_{j_1}, t'_{j_1}, \dots, z'_{j_{s-1}}, t'_{j_{s-1}}$ be letters that do not occur in $\mathbf{p}_1\mathbf{q}_1\mathbf{r}_1x$. Put $\mathbf{p}_2 = z'_{j_1}t'_{j_1} \cdots z'_{j_{s-1}}t'_{j_{s-1}}\mathbf{p}_1$. Further, we denote by $\mathbf{q}_2$ the word obtained from $\mathbf{q}_1$ by replacing $\mathbf{v}_1$ with $z_{j_1}z'_{j_1} \cdots z_{j_{s-1}}z'_{j_{s-1}}z_{j_s}$. Finally, we put $\mathbf{r}_2 = \mathbf{r}_1$. The identity $\mathbf{p}_1x\mathbf{q}_1x\mathbf{r}_1 \approx \mathbf{p}_1x^2\mathbf{q}_1\mathbf{r}_1$ follows from $\mathbf{p}_2x\mathbf{q}_2x\mathbf{r}_2 \approx \mathbf{p}_2x^2\mathbf{q}_2\mathbf{r}_2$ by substitution of 1 for z'_{j_1}, t'_{j_1}, …, $z'_{j_{s-1}}$, $t'_{j_{s-1}}$.

We continue this process and apply analogous modifications of our identity with the use of the words $\mathbf{u}_2, \mathbf{v}_2, \dots, \mathbf{u}_k, \mathbf{v}_k$. As a result, we obtain an identity of the form

$$\mathbf{p}_{2k}x\mathbf{q}_{2k}x\mathbf{r}_{2k} \approx \mathbf{p}_{2k}x^2\mathbf{q}_{2k}\mathbf{r}_{2k}, \tag{4.13}$$

which implies an identity of the form (4.11) fixed at the beginning of the proof. We can evidently rename the letters and assume that $\mathbf{p}_{2k} = z_1t_1 \cdots z_pt_p$, $\mathbf{q}_{2k} = z_{\xi(1)} \cdots z_{\xi(p+q)}$ and $\mathbf{r}_{2k} = t_{p+1}z_{p+1} \cdots t_{p+q}z_{p+q}$ for some natural numbers p, q and some permutation $\xi \in S_{p+q}$ with $\xi(i) \le p$ for all odd i and $\xi(i) > p$ for all even i. It remains to verify that $p = q$. For $i = 1, \dots, k$, we denote the length of $\mathbf{u}_i$ by n_i and the length of $\mathbf{v}_i$ by m_i. Then $n_1 + \cdots + n_k = n$ and $m_1 + \cdots + m_k = m$. It is easy to see that

$$\begin{aligned} p &= n + (m_1 - 1) + \cdots + (m_k - 1) = n + m - k \\ &= m + n - k = m + (n_1 - 1) + \cdots + (n_k - 1) = q. \end{aligned}$$

Therefore, the identity (4.13) has the form (4.12). ∎

Lemma 4.5. *Suppose that a monoid variety* $\mathbf{X}$ *satisfies the identities*

$$xyxzx \approx x^2yz, \tag{4.14}$$

$$x^2y \approx yx^2 \tag{4.15}$$

and (4.11) *for all* n, m *and* $\theta \in S_{n+m}$. *Let* $\mathbf{u}$ *be a word. If there is a letter* $x \in \mathrm{mul}(\mathbf{u})$ *such that* $\mathbf{u}(x, y) \ne xyx$ *for any letter* y *then* $\mathbf{X}$ *satisfies the identity*

$$\mathbf{u} \approx x^2\mathbf{u}_x. \tag{4.16}$$

Proof. Suppose first that $\mathrm{occ}_x(\mathbf{u}) > 2$. Then $\mathbf{u} = \mathbf{u}_1x\mathbf{u}_2x\mathbf{u}_3 \cdots \mathbf{u}_nx\mathbf{u}_{n+1}$ where $n > 2$ and $\mathbf{u}_1, \dots, \mathbf{u}_{n+1}$ are possibly empty words with $x \notin \mathrm{con}(\mathbf{u}_1 \cdots \mathbf{u}_{n+1})$. Clearly, $\mathbf{u}_1 \cdots \mathbf{u}_{n+1} = \mathbf{u}_x$. Then $\mathbf{X}$ satisfies the identities

$$\mathbf{u} = \mathbf{u}_1x\mathbf{u}_2x\mathbf{u}_3 \cdots \mathbf{u}_nx\mathbf{u}_{n+1} \stackrel{(4.14)}{\approx} \mathbf{u}_1x^2\mathbf{u}_2\mathbf{u}_3 \cdots \mathbf{u}_{n+1} \stackrel{(4.15)}{\approx} x^2\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_{n+1} = x^2\mathbf{u}_x,$$

whence (4.16) holds in $\mathbf{X}$.

It remains to consider the case when $\mathrm{occ}_x(\mathbf{u}) = 2$. Then $\mathbf{u} = \mathbf{u}_1x\mathbf{u}_2x\mathbf{u}_3$ and $x \notin \mathrm{con}(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3)$. If $\mathbf{u}_2 = \lambda$ then $\mathbf{u} = \mathbf{u}_1x^2\mathbf{u}_3 \stackrel{(4.15)}{\approx} x^2\mathbf{u}_1\mathbf{u}_3 = x^2\mathbf{u}_x$ hold in $\mathbf{X}$, and we are done. Let now $\mathbf{u}_2 \ne \lambda$.

If $y \in \mathrm{con}(\mathbf{u}_2)$ and $y \in \mathrm{sim}(\mathbf{u})$ then $\mathbf{u}(x, y) = xyx$, a contradiction. Thus, $y \in \mathrm{mul}(\mathbf{u})$ for any $y \in \mathrm{con}(\mathbf{u}_2)$. Suppose that $\mathrm{occ}_y(\mathbf{u}) > 2$ for some $y \in \mathrm{con}(\mathbf{u}_2)$. Then we can use the same arguments as in the first paragraph of the proof to conclude that $\mathbf{X}$ satisfies $\mathbf{u} \approx y^2\mathbf{u}_y$. This identity can be rewritten in the form $\mathbf{u} \approx \mathbf{u}'_1x\mathbf{u}'_2x\mathbf{u}'_3$ where $\mathbf{u}'_1 = y^2\mathbf{u}_1$, $\mathbf{u}'_2 = (\mathbf{u}_2)_y$ and $\mathbf{u}'_3 = (\mathbf{u}_3)_y$. Thus, we can remove from $\mathbf{u}_2$ all letters y with $\mathrm{occ}_y(\mathbf{u}) > 2$. In other words, we can assume that either $\mathbf{u}_2 = \lambda$ or $\mathrm{occ}_y(\mathbf{u}) = 2$ for all $y \in \mathrm{con}(\mathbf{u}_2)$.

The former case has already been considered in the previous paragraph. Now we examine the latter case.

Recall that a word $\mathbf{w}$ is called *linear* if $\operatorname{occ}_x(\mathbf{w}) \le 1$ for any letter x. Suppose that $\mathbf{u}_2$ is linear, say, $\mathbf{u}_2 = y_1 \cdots y_k$ for some letters $y_1, \dots, y_k$. Then either $y_i \in \operatorname{con}(\mathbf{u}_1) \setminus \operatorname{con}(\mathbf{u}_3)$ or $y_i \in \operatorname{con}(\mathbf{u}_3) \setminus \operatorname{con}(\mathbf{u}_1)$ for any $1 \le i \le k$. Renaming the letters $y_1, \dots, y_k$ if necessary, we may assume that $y_1, \dots, y_n \in \operatorname{con}(\mathbf{u}_1) \setminus \operatorname{con}(\mathbf{u}_3)$ and $y_{n+1}, \dots, y_{n+m} \in \operatorname{con}(\mathbf{u}_3) \setminus \operatorname{con}(\mathbf{u}_1)$ for some n and m with $n + m = k$. Then

$$\mathbf{u} = \mathbf{u}_1 x y_{\theta(1)} y_{\theta(2)} \cdots y_{\theta(n+m)} x \mathbf{u}_3$$

for some $\theta \in S_{n+m}$. We also have

$$\mathbf{u}_1 = \mathbf{w}_0 y_1 \mathbf{w}_1 y_2 \mathbf{w}_2 \cdots y_n \mathbf{w}_n \quad \text{and} \quad \mathbf{u}_3 = \mathbf{w}_{n+1} y_{n+1} \mathbf{w}_{n+2} y_{n+2} \cdots \mathbf{w}_{n+m} y_{n+m} \mathbf{w}_{n+m+1}$$

for some possibly empty words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{n+m+1}$. Then $\mathbf{X}$ satisfies the identities

$$\begin{aligned}
\mathbf{u} &= \mathbf{w}_0 \Big(\prod_{i=1}^{n} y_i \mathbf{w}_i\Big) x \Big(\prod_{i=1}^{n+m} y_{\theta(i)}\Big) x \Big(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Big) \mathbf{w}_{n+m+1} \\
&\stackrel{(4.11)}{\approx} \mathbf{w}_0 \Big(\prod_{i=1}^{n} y_i \mathbf{w}_i\Big) x^2 \Big(\prod_{i=1}^{n+m} y_{\theta(i)}\Big) \Big(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Big) \mathbf{w}_{n+m+1} \\
&\stackrel{(4.15)}{\approx} x^2 \mathbf{w}_0 \Big(\prod_{i=1}^{n} y_i \mathbf{w}_i\Big) \Big(\prod_{i=1}^{n+m} y_{\theta(i)}\Big) \Big(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Big) \mathbf{w}_{n+m+1} \\
&= x^2 \mathbf{u}_x.
\end{aligned}$$

We see that $\mathbf{X}$ satisfies the identity (4.16) again.

It remains to consider the case when $\mathbf{u}_2$ is not linear. Then there is a letter $y \in \operatorname{con}(\mathbf{u}_2)$ such that $\mathbf{u}_2 = \mathbf{v}_1 y \mathbf{v}_2 y \mathbf{v}_3$ where $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are possibly empty words, $y \notin \operatorname{con}(\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3)$ and $\mathbf{v}_2$ is either empty or linear. If $\mathbf{v}_2$ is linear then the same arguments as in the previous paragraph show that

$$\mathbf{u} = \mathbf{u}_1 x \mathbf{v}_1 y \mathbf{v}_2 y \mathbf{v}_3 x \mathbf{u}_3 \approx y^2 \mathbf{u}_y = y^2 \mathbf{u}_1 x \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 x \mathbf{u}_3 = \mathbf{u}_1' x \mathbf{u}_2' x \mathbf{u}_3$$

hold in $\mathbf{X}$ where $\mathbf{u}_1' = y^2 \mathbf{u}_1$ and $\mathbf{u}_2' = \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3$. If $\mathbf{v}_2 = \lambda$ then

$$\mathbf{u} = \mathbf{u}_1 x \mathbf{v}_1 y^2 \mathbf{v}_3 x \mathbf{u}_3 \stackrel{(4.15)}{\approx} y^2 \mathbf{u}_1 x \mathbf{v}_1 \mathbf{v}_3 x \mathbf{u}_3 = \mathbf{u}_1' x \mathbf{u}_2' x \mathbf{u}_3$$

is valid in $\mathbf{X}$ where $\mathbf{u}_1' = y^2 \mathbf{u}_1$ and $\mathbf{u}_2' = \mathbf{v}_1 \mathbf{v}_3$. In both the cases, $y \notin \operatorname{con}(\mathbf{u}_2')$. In other words, we can remove the letter y from $\mathbf{u}_2$. Further, we can repeat these arguments as long as the word $\mathbf{u}_2$ is non-empty and non-linear. In other words, we may assume that $\mathbf{u}_2$ is either empty or linear. Both these cases have already been considered above. Thus, we have proved that $\mathbf{X}$ always satisfies (4.16). ∎

Lemma 4.6. $\mathbf{L} = \operatorname{var} S(xzxyty)$.

Proof. Put $\mathbf{Z} = \operatorname{var} S(xzxyty)$. First, we verify that $\mathbf{Z} \subseteq \mathbf{L}$. In view of Lemma 2.3, it suffices to check that the word $xzxyty$ is an isoterm for $\mathbf{L}$. Put

$$\Psi = \{x^2 y \approx y x^2,\ xyxzx \approx x^2 yz,\ \sigma_1,\ \sigma_2,\ \mathbf{w}_n(\pi, \tau) \approx \mathbf{w}_n'(\pi, \tau) \mid n \in \mathbb{N},\ \pi, \tau \in S_n\}.$$

We recall that $\mathbf{L} = \operatorname{var} \Psi$. We suppose that $\mathbf{L}$ satisfies a non-trivial identity $xzxyty \approx \mathbf{w}$ for some word $\mathbf{w}$. Therefore, there exists a *deduction* of the identity $xzxyty \approx \mathbf{w}$ from the identity system Ψ, i.e., a sequence of words

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \tag{4.17}$$

such that $\mathbf{v}_0 = xzxyty$, $\mathbf{v}_m = \mathbf{w}$ and, for any $0 \le i < m$, there exist words $\mathbf{a}_i$, $\mathbf{b}_i$, an identity $\mathbf{s}_i \approx \mathbf{t}_i \in \Psi$ and an endomorphism ξ_i of F^1 such that either $\mathbf{v}_i = \mathbf{a}_i\xi_i(\mathbf{s}_i)\mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i\xi_i(\mathbf{t}_i)\mathbf{b}_i$, or $\mathbf{v}_i = \mathbf{a}_i\xi_i(\mathbf{t}_i)\mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i\xi_i(\mathbf{s}_i)\mathbf{b}_i$. We can assume without loss of generality that (4.17) is the shortest such deduction. In particular, this means that $xzxyty \ne \mathbf{v}_1$. We note that if $\xi_0(x) = \lambda$ then $\xi_0(\mathbf{s}_0) = \xi_0(\mathbf{t}_0)$ for any $\mathbf{s}_0 \approx \mathbf{t}_0 \in \Psi$. The last equality implies that $xzxyty = \mathbf{v}_1$, but this is impossible. Thus, we can assume that $\xi_0(x) \ne \lambda$.

Suppose that $xzxyty = \mathbf{v}_0 = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0$ and $\mathbf{v}_1 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0$. The case when $\mathbf{s}_0 = x^2y$ is impossible because $\xi_0(\mathbf{s}_0)$ contains the square of a non-empty word, while $xzxyty$ is square-free. The case when $\mathbf{s}_0 = xyxzx$ is also impossible because there is a letter that occurs in $\xi_0(\mathbf{s}_0)$ at least three times, while every letter from $\mathrm{con}(xzxyty)$ occurs in the word $xzxyty$ no more than twice. Finally, the case when $\mathbf{s}_0 = \mathbf{w}_n(\pi, \tau)$ for some $n \in \mathbb{N}$, $\pi, \tau \in S_n$ is impossible because there exists a letter $c \in \xi_0(x)$ such that c is multiple in $\xi_0(\mathbf{s}_0)$ and every letter located between the first and the second occurrences of c in $\xi_0(\mathbf{s}_0)$ is multiple, while for every $d \in \mathrm{mul}(xzxyty)$ there is a letter $e \in \mathrm{sim}(xzxyty)$ such that e lies between the first and the second occurrences of d in $xzxyty$. So, the identity $\mathbf{s}_0 \approx \mathbf{t}_0$ is either σ_1 or σ_2. By symmetry, we can consider only the first case when $\mathbf{s}_0 \approx \mathbf{t}_0$ is equal to σ_1. Then $\mathbf{s}_0 = xyzxty$ and $\mathbf{t}_0 = yxzxty$. Since $\xi_0(x) \ne \lambda$, we see that $\mathrm{con}(\xi_0(x))$ contains a letter a. Then $a \in \{x, y\}$ because $a \in \mathrm{mul}(\xi_0(\mathbf{s}_0))$. Suppose that $a = x$. Then $\xi_0(y) = \lambda$ because

$$xzxyty = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0 = \mathbf{a}_0\xi_0(x)\xi_0(y)\xi_0(z)\xi_0(x)\xi_0(t)\xi_0(y)\mathbf{b}_0.$$

Therefore, $\xi_0(\mathbf{t}_0) = \xi_0(x)\xi_0(z)\xi_0(x)\xi_0(t) = \xi_0(\mathbf{s}_0)$. Then

$$\mathbf{v}_1 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0 = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0 = xzxyty,$$

contradicting the choice of (4.17). The case $a = y$ is handled similarly.

Suppose now that $xzxyty = \mathbf{v}_0 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0$. The case when

$$\mathbf{t}_0 \in \{yx^2,\ x^2yz,\ \mathbf{w}_n'(\pi, \tau) \mid n \in \mathbb{N},\ \pi, \tau \in S_n\}$$

is impossible because $\xi_0(\mathbf{t}_0)$ contains the square of a non-empty word in this case, while $xzxyty$ is square-free. So, the identity $\mathbf{s}_0 \approx \mathbf{t}_0$ is either σ_1 or σ_2. Arguments similar to those from the previous paragraph yield a contradiction with the fact that $xzxyty$ and $\mathbf{v}_1$ are distinct.

Thus, we have verified that $xzxyty$ is an isoterm for $\mathbf{L}$, and therefore $\mathbf{Z} \subseteq \mathbf{L}$. It remains to verify the opposite inclusion. Suppose that $\mathbf{Z}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$. We need to prove that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{L}$. Lemma 4.4 allows us to use Lemma 4.5. Let x be a letter multiple in $\mathbf{u}$ and $\mathbf{u}(x, y) \ne xyx$ for any letter y. By Lemma 4.5, the variety $\mathbf{L}$ satisfies the identity (4.16). Obviously, $\mathbf{C}_2 \subseteq \mathbf{Z}$, whence $x \in \mathrm{mul}(\mathbf{v})$ by Proposition 2.2. Since $xzxyty$ is an isoterm for $\mathbf{Z}$, xyx is an isoterm for $\mathbf{Z}$ too. Therefore, $\mathbf{v}(x, y) \ne xyx$ for any letter y. Lemma 4.5 again shows that the identity $\mathbf{v} \approx x^2\mathbf{v}_x$ holds in $\mathbf{L}$. Thus, if $\mathbf{u}_x \approx \mathbf{v}_x$ holds in $\mathbf{L}$ then this variety satisfies $\mathbf{u} \approx x^2\mathbf{u}_x \approx x^2\mathbf{v}_x \approx \mathbf{v}$. So, we can remove from $\mathbf{u} \approx \mathbf{v}$ all multiple letters x such that $\mathbf{u}(x, y) \ne xyx$ for any y. In other words, we may assume that for any $x \in \mathrm{mul}(\mathbf{u})$ there is a letter y such that $\mathbf{u}(x,y) = xyx = \mathbf{v}(x,y)$. In particular, $\mathrm{occ}_x(\mathbf{u}), \mathrm{occ}_x(\mathbf{v}) \le 2$ for any letter x.

Lemma 2.1 and the evident inclusion $\mathbf{C}_2 \subseteq \mathbf{Z}$ imply that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. It is clear that for any $a, b \notin \mathrm{con}(\mathbf{u})$, the identities $\mathbf{u} \approx \mathbf{v}$ and $a\mathbf{u}b \approx a\mathbf{v}b$ are equivalent in the class of monoids. Therefore, we can assume that the first and the last letters in each of the words $\mathbf{u}$ and $\mathbf{v}$ are simple in that word. Let $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v}) = \{t_0, t_1, \dots, t_m\}$. We can assume that $\mathbf{v}(t_1, \dots, t_m) = t_1 \cdots t_m$. In view of Lemma 2.7, $\mathbf{D}_1 \subseteq \mathbf{Z}$. Then Proposition 2.13 implies that

$$\mathbf{u} = t_0\mathbf{a}_1t_1\mathbf{a}_2t_2 \cdots t_{m-1}\mathbf{a}_mt_m \quad \text{and} \quad \mathbf{v} = t_0\mathbf{b}_1t_1\mathbf{b}_2t_2 \cdots t_{m-1}\mathbf{b}_mt_m$$

for some possibly empty words $\mathbf{a}_1, \dots, \mathbf{a}_m$ and $\mathbf{b}_1, \dots, \mathbf{b}_m$.

Let $0 \le i \le m-1$. Then $\mathbf{u} = \mathbf{w}_1t_i\mathbf{a}_{i+1}t_{i+1}\mathbf{w}_2$ where

$$\mathbf{w}_1 = \begin{cases} t_0\mathbf{a}_1t_1 \cdots t_{i-1}\mathbf{a}_i & \text{if } 0 < i \le m-1, \\ \lambda & \text{if } i = 0, \end{cases}$$

$$\mathbf{w}_2 = \begin{cases} \mathbf{a}_{i+2}t_{i+2} \cdots \mathbf{a}_mt_m & \text{if } 0 \le i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

We are going to check that

$$\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{u}_1'\mathbf{u}_2\mathbf{u}_2' \cdots \mathbf{u}_k\mathbf{u}_k', \tag{4.18}$$

and therefore $\mathbf{u} = \mathbf{w}_1t_i\mathbf{u}_1\mathbf{u}_1'\mathbf{u}_2\mathbf{u}_2' \cdots \mathbf{u}_k\mathbf{u}_k't_{i+1}\mathbf{w}_2$ for some possibly empty words $\mathbf{u}_1, \mathbf{u}_k'$ and non-empty words $\mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \dots, \mathbf{u}_k$ such that $\mathrm{con}(\mathbf{u}_j) \subseteq \mathrm{con}(\mathbf{w}_1)$ and $\mathrm{con}(\mathbf{u}_j') \subseteq \mathrm{con}(\mathbf{w}_2)$ for all $j = 1, \dots, k$. If $\mathbf{a}_{i+1} = \lambda$ then (4.18) holds with $k = 1$ and $\mathbf{u}_1 = \mathbf{u}_1' = \lambda$. Suppose now that $\mathbf{a}_{i+1} \ne \lambda$. Let $x \in \mathrm{con}(\mathbf{a}_{i+1})$. Then $x \in \mathrm{mul}(\mathbf{u})$. There is $y \in \mathrm{sim}(\mathbf{u})$ with $\mathbf{u}(x,y) = xyx$. Suppose that $x \in \mathrm{mul}(\mathbf{a}_{i+1})$. Then xyx is a subword of $\mathbf{a}_{i+1}$. This means that y is simple in $\mathbf{a}_{i+1}$. But this is not the case because $y \ne t_j$ for any $0 \le j \le m$. Thus, x is simple in $\mathbf{a}_{i+1}$, whence $x \in \mathrm{con}(\mathbf{w}_1\mathbf{w}_2)$. We have proved that every letter from $\mathrm{con}(\mathbf{a}_{i+1})$ is simple in $\mathbf{a}_{i+1}$ and occurs either in $\mathbf{w}_1$ or in $\mathbf{w}_2$.

Let $\mathbf{u}_1$ be the maximal prefix of $\mathbf{a}_{i+1}$ such that $\mathrm{con}(\mathbf{u}_1) \subseteq \mathrm{con}(\mathbf{w}_1)$ (if the first letter of $\mathbf{a}_{i+1}$ does not occur in $\mathbf{w}_1$ then $\mathbf{u}_1 = \lambda$). Then $\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{b}$ for some possibly empty word $\mathbf{b}$. If $\mathbf{b} = \lambda$ then (4.18) holds with $k = 1$ and $\mathbf{u}_1' = \lambda$. Otherwise, let $\mathbf{u}_1'$ be the maximal prefix of $\mathbf{b}$ such that $\mathrm{con}(\mathbf{u}_1') \subseteq \mathrm{con}(\mathbf{w}_2)$. Then $\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{u}_1'\mathbf{c}$ for some possibly empty word $\mathbf{c}$. If $\mathbf{c} = \lambda$ then (4.18) holds with $k = 1$. Otherwise, let $\mathbf{u}_2$ be the maximal prefix of $\mathbf{c}$ such that $\mathrm{con}(\mathbf{u}_2) \subseteq \mathrm{con}(\mathbf{w}_1)$. Continuing this process, we obtain (4.18).

Put

$$\mathbf{w}_1' = \begin{cases} t_0\mathbf{b}_1t_1 \cdots t_{i-1}\mathbf{b}_i & \text{if } 0 < i \le m-1, \\ \lambda & \text{if } i = 0, \end{cases}$$

$$\mathbf{w}_2' = \begin{cases} \mathbf{b}_{i+2}t_{i+2} \cdots \mathbf{b}_mt_m & \text{if } 0 \le i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

The same arguments as above show that $\mathbf{b}_{i+1} = \mathbf{v}_1\mathbf{v}_1'\mathbf{v}_2\mathbf{v}_2' \cdots \mathbf{v}_r\mathbf{v}_r'$ for some natural r, possibly empty words $\mathbf{v}_1, \mathbf{v}_r'$ and non-empty words $\mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_2', \dots, \mathbf{v}_r$ such that

$\mathrm{con}(\mathbf{v}_j) \subseteq \mathrm{con}(\mathbf{w}_1')$ and $\mathrm{con}(\mathbf{v}_j') \subseteq \mathrm{con}(\mathbf{w}_2')$ for all $j = 1, \ldots, r$. Therefore,

$$\mathbf{v} = \mathbf{w}_1' t_i \mathbf{v}_1 \mathbf{v}_1' \mathbf{v}_2 \mathbf{v}_2' \cdots \mathbf{v}_r \mathbf{v}_r' t_{i+1} \mathbf{w}_2'.$$

Further, we may assume that $k \ge r$. We are going to verify that $k = r$, $\mathrm{con}(\mathbf{u}_j) = \mathrm{con}(\mathbf{v}_j)$ and $\mathrm{con}(\mathbf{u}_j') = \mathrm{con}(\mathbf{v}_j')$ for all $j = 1, \ldots, r$.

Let $x \in \mathrm{con}(\mathbf{u}_1)$. As shown above, $\mathbf{u}(x, t_i) = xt_ix$. Therefore, $\mathbf{v}(x, t_i) = xt_ix$ too, whence $\mathrm{occ}_x(\mathbf{w}_1') = 1$. Note that $\mathbf{v}(x, t_{i+1}) \ne xt_{i+1}x$ because $\mathbf{u}(x, t_{i+1}) = x^2t_{i+1}$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_2')$, whence $x \in \mathrm{con}(\mathbf{v}_1 \cdots \mathbf{v}_r)$. If $x \notin \mathrm{con}(\mathbf{v}_1)$ then $x \in \mathrm{con}(\mathbf{v}_p)$ for some $p > 1$. Then there exists $y \in \mathrm{con}(\mathbf{v}_{p-1}')$. Note that $\mathbf{u}(y, t_{i+1}) = \mathbf{v}(y, t_{i+1}) = yt_{i+1}y$. Therefore, $y \in \mathrm{con}(\mathbf{w}_2)$, whence $y \in \mathrm{con}(\mathbf{u}_j')$ for some $1 \le j \le k$. Then $\mathbf{u}(x, y, t_i, t_{i+1}) = xt_ixyt_{i+1}y$, while $\mathbf{v}(x, y, t_i, t_{i+1}) = xt_iyxt_{i+1}y$. This contradicts $xt_ixyt_{i+1}y$ being an isoterm for $\mathbf{Z}$. Thus, $x \in \mathrm{con}(\mathbf{v}_1)$, whence $\mathrm{con}(\mathbf{u}_1) \subseteq \mathrm{con}(\mathbf{v}_1)$. Analogously, $\mathrm{con}(\mathbf{v}_1) \subseteq \mathrm{con}(\mathbf{u}_1)$. Therefore, $\mathrm{con}(\mathbf{u}_1) = \mathrm{con}(\mathbf{v}_1)$.

Let $x \in \mathrm{con}(\mathbf{u}_1')$. As shown above, $\mathbf{u}(x, t_{i+1}) = xt_{i+1}x$. Therefore, $\mathbf{v}(x, t_{i+1}) = xt_{i+1}x$ too, whence $\mathrm{occ}_x(\mathbf{w}_2') = 1$. Note that $\mathbf{v}(x, t_i) \ne xt_ix$ because $\mathbf{u}(x, t_i) = t_ix^2$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_1')$, whence $x \in \mathrm{con}(\mathbf{v}_1' \cdots \mathbf{v}_r')$. If $x \notin \mathrm{con}(\mathbf{v}_1')$ then $x \in \mathrm{con}(\mathbf{v}_p')$ for some $p > 1$. Then there exists $y \in \mathrm{con}(\mathbf{v}_p)$. Note that $\mathbf{u}(y, t_i) = \mathbf{v}(y, t_i) = yt_iy$. Therefore, $y \in \mathrm{con}(\mathbf{w}_1)$, whence $y \in \mathrm{con}(\mathbf{u}_j)$ for some $1 \le j \le k$. Note that $y \notin \mathrm{con}(\mathbf{u}_1)$. Indeed, if $y \in \mathrm{con}(\mathbf{u}_1)$ then $y \in \mathrm{con}(\mathbf{v}_1)$ because $\mathrm{con}(\mathbf{u}_1) = \mathrm{con}(\mathbf{v}_1)$. Hence $\mathrm{occ}_y(\mathbf{v}) \ge \mathrm{occ}_y(\mathbf{v}_1\mathbf{v}_p\mathbf{w}_2') \ge 3$, a contradiction. So, $y \in \mathrm{con}(\mathbf{u}_j)$ for some $2 \le j \le k$. Then $\mathbf{u}(x, y, t_i, t_{i+1}) = xt_ixyt_{i+1}y$, while $\mathbf{v}(x, y, t_i, t_{i+1}) = xt_iyxt_{i+1}y$. This contradicts $xt_ixyt_{i+1}y$ being an isoterm for $\mathbf{Z}$. Thus, $x \in \mathrm{con}(\mathbf{v}_1')$, whence $\mathrm{con}(\mathbf{u}_1') \subseteq \mathrm{con}(\mathbf{v}_1')$. Analogously, $\mathrm{con}(\mathbf{v}_1') \subseteq \mathrm{con}(\mathbf{u}_1')$. We have proved that $\mathrm{con}(\mathbf{u}_1') = \mathrm{con}(\mathbf{v}_1')$.

Repeating the arguments from the previous two paragraphs with evident modifications, we can check that $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ and $\mathrm{con}(\mathbf{u}_i') = \mathrm{con}(\mathbf{v}_i')$ for $i = 2, \ldots, r$.

If $k > r$ then there is a letter $x \in \mathrm{con}(\mathbf{u}_{r+1})$. As shown above, $\mathbf{u}(x, t_i) = xt_ix$. Therefore, $\mathbf{v}(x, t_i) = xt_ix$ too, whence $\mathrm{occ}_x(\mathbf{w}_1') = 1$. Note also that $\mathbf{v}(x, t_{i+1}) = \mathbf{u}(x, t_{i+1}) = x^2t_{i+1}$. In particular, $\mathbf{v}(x, t_{i+1}) \ne xt_{i+1}x$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_2')$, whence $x \in \mathrm{con}(\mathbf{v}_1 \cdots \mathbf{v}_r)$. Then $x \in \mathrm{con}(\mathbf{u}_1 \cdots \mathbf{u}_r)$ because $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ for $i = 1, \ldots, r$. Thus, $\mathrm{occ}_x(\mathbf{u}) \ge \mathrm{occ}_x(\mathbf{w}_1\mathbf{u}_1 \cdots \mathbf{u}_{r+1}) \ge 3$, a contradiction. Therefore, $k = r$.

We have proved that $k = r$, $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ and $\mathrm{con}(\mathbf{u}_i') = \mathrm{con}(\mathbf{v}_i')$ for all $i = 1, \ldots, k$. Fix $s \in \{1, \ldots, k\}$. Then $\mathbf{u}_s$ and $\mathbf{v}_s$ are linear words depending on the same letters. The same is true for $\mathbf{u}_s'$ and $\mathbf{v}_s'$. The identity σ_1 [respectively σ_2] allows us to swap the first [the second] occurrences of two multiple letters whenever these occurrences are adjacent to each other. Therefore, the identities σ_1 and σ_2 allow us to reorder letters within $\mathbf{u}_s$ and $\mathbf{u}_s'$ in an arbitrary way. Thus, if we replace $\mathbf{u}_s$ by $\mathbf{v}_s$ and $\mathbf{u}_s'$ by $\mathbf{v}_s'$ in $\mathbf{u}$ then the word we obtain should be equal to $\mathbf{u}$ in $\mathbf{L}$. This is true for all $s = 1, \ldots, k$. Hence $\mathbf{L}$ satisfies the identities

$$\begin{aligned}\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2 &= \mathbf{w}_1 t_i \mathbf{u}_1 \mathbf{u}_1' \mathbf{u}_2 \mathbf{u}_2' \cdots \mathbf{u}_k \mathbf{u}_k' t_{i+1} \mathbf{w}_2 \\ &\approx \mathbf{w}_1 t_i \mathbf{v}_1 \mathbf{v}_1' \mathbf{v}_2 \mathbf{v}_2' \cdots \mathbf{v}_k \mathbf{v}_k' t_{i+1} \mathbf{w}_2 = \mathbf{w}_1 t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2.\end{aligned}$$

Thus, if we replace $\mathbf{a}_{i+1}$ by $\mathbf{b}_{i+1}$ in $\mathbf{u}$ then the resulting word should be equal to $\mathbf{u}$ in $\mathbf{L}$.

This is true for all $i = 0, \ldots, m-1$. Therefore, $\mathbf{L}$ satisfies the identities

$$\mathbf{u} = t_0\mathbf{a}_1t_1\mathbf{a}_2t_2\cdots t_{m-1}\mathbf{a}_mt_m \approx t_0\mathbf{b}_1t_1\mathbf{b}_2t_2\cdots t_{m-1}\mathbf{b}_mt_m = \mathbf{v}.$$

The lemma is proved. ∎

Lemma 4.6 and [8, Lemma 5.10] imply that any proper subvariety of $\mathbf{L}$ is contained in $\operatorname{var} S(xyx)$. Lemmas 2.7 and 2.8 now imply

Corollary 4.7. *The lattice* $L(\mathbf{L})$ *is the chain* $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \mathbf{L}$. ∎

A non-finitely based variety all whose proper subvarieties are finitely based is called *limit*. The variety $\operatorname{var} S(xzxyty)$ is limit by [8, Proposition 5.1]. Thus, Lemma 4.6 implies

Corollary 4.8. *The variety* $\mathbf{L}$ *is a limit variety. In particular, it does not have a finite basis of identities.* ∎

According to the result of [11] mentioned in Chapter 1, there are uncountably many periodic group varieties whose subvariety lattice is the 3-element chain. Let $\mathcal{G}$ be the class of all such varieties. Since the class of finitely based group varieties is countably infinite, the class $\mathcal{G}$ contains non-finitely based varieties. Group varieties whose subvariety lattice is the 2-element chain are varieties of Abelian groups of a prime exponent. They are finitely based. Thus, all non-finitely based varieties from $\mathcal{G}$ are limit varieties. But explicit examples of limit chain group varieties have not been published so far.

We denote by $\mathbf{M}$ the subvariety of $\mathbf{N}$ given within $\mathbf{N}$ by the identity

$$\alpha_1 : \; x_1y_1x_0x_1y_1 \approx y_1x_1x_0x_1y_1.$$

Note that α_1 belongs to a countably infinite series of identities α_k that will be defined in Section 6.1.

Lemma 4.9. *Let* $\mathbf{X}$ *be a monoid variety and* $\mathbf{D}_2 \subseteq \mathbf{X}$.

(i) *If* $\mathbf{L} \nsubseteq \mathbf{X}$ *then* $\mathbf{X}$ *satisfies the identity* γ_1.
(ii) *If* $\mathbf{M} \nsubseteq \mathbf{X}$ *then* $\mathbf{X}$ *satisfies the identity* σ_1.

Proof. (i) According to Lemmas 2.3 and 4.6, the variety $\mathbf{X}$ satisfies a non-trivial identity of the form $xzxyty \approx \mathbf{w}$. Note that xyx is an isoterm for $\mathbf{X}$ by Lemmas 2.3 and 2.7. Then [20, Fact 4.1(i)] implies that $\mathbf{w} = xzyxty$. Therefore, γ_1 holds in $\mathbf{X}$.

(ii) According to Lemmas 2.3 and 2.7, xyx is an isoterm for $\mathbf{X}$. Further, the variety $\mathbf{M}$ is generated by the monoid $S(xyzxty)$ (this fact is dual to Proposition 1 in Erratum to [8]). Then $S(xyzxty) \notin \mathbf{X}$, whence $\mathbf{X}$ satisfies a non-trivial identity of the form $xyzxty \approx \mathbf{w}$ by Lemma 2.3. Fact 4.1(ii) of [20] implies that $\mathbf{w} = yxzxty$. Therefore, σ_1 holds in $\mathbf{X}$. ∎

We return to the examination of a chain variety $\mathbf{V}$. In Section 4.1 we have reduced the considerations to the case when $\mathbf{D}_2 \subseteq \mathbf{V}$. Then $\mathbf{E} \nsubseteq \mathbf{V}$ because $\mathbf{D}_2$ and $\mathbf{E}$ are non-comparable. The variety $\mathbf{V}$ satisfies (4.6) by Lemma 4.3. Similarly, the fact that $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$ implies that $\mathbf{V}$ satisfies (4.7) by the dual of Lemma 4.3. Hence (4.15) holds in $\mathbf{V}$. If $\mathbf{V}$ contains neither $\mathbf{L}$, $\mathbf{M}$ nor $\overleftarrow{\mathbf{M}}$ then Lemma 4.9 and the dual of its claim (ii) imply that $\mathbf{V}$ satisfies σ_1, σ_2 and γ_1, whence $\mathbf{V} \subseteq \mathbf{D}$.

It remains to consider the case when $\mathbf{V}$ contains $\mathbf{L}$, $\mathbf{M}$ or $\overleftarrow{\mathbf{M}}$. Then $\mathbf{V}$ does not contain $\mathbf{D}_3$ because $\mathbf{L}$, $\mathbf{M}$ and $\overleftarrow{\mathbf{M}}$ are non-comparable with $\mathbf{D}_3$. Lemma 2.15 and the fact that $\mathbf{V}$ satisfies (4.15) imply that (4.14) holds in $\mathbf{V}$.

Let $\mathbf{M} \subseteq \mathbf{V}$. Then $\mathbf{V}$ contains neither $\mathbf{L}$ nor $\overleftarrow{\mathbf{M}}$. Lemma 4.9(i) and the dual of Lemma 4.9(ii) imply that $\mathbf{V} \subseteq \mathbf{N}$. Dual arguments show that if $\overleftarrow{\mathbf{M}} \subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \overleftarrow{\mathbf{N}}$.

We have reduced our considerations to the case when $\mathbf{L} \subseteq \mathbf{V}$.

4.3. The case when $\mathbf{L} \subseteq \mathbf{V}$. Clearly, here $\mathbf{M}, \overleftarrow{\mathbf{M}} \nsubseteq \mathbf{V}$. Lemma 4.9(ii) and its dual imply that $\mathbf{V}$ satisfies the identities σ_1 and σ_2. As we have already seen above, $\mathbf{V}$ satisfies (4.14) and (4.15) as well. Thus, $\mathbf{V}$ is contained in

$$\mathbf{O} = \operatorname{var}\{x^2y \approx yx^2,\ xyxzx \approx x^2yz,\ \sigma_1,\ \sigma_2\}.$$

To complete the proof of the necessity in Theorem 1.1, it suffices to verify that $\mathbf{V} \subseteq \mathbf{L}$. To this end, it remains to check that $\mathbf{V}$ satisfies all identities of the form (4.12) where n is a natural number and $\pi, \tau \in S_n$. To do this, we need several auxiliary claims. Let $n \in \mathbb{N}$, $0 \le k \le \ell \le n$ and $\pi, \tau \in S_n$. Put

$$\mathbf{w}_n^{k,\ell}(\pi,\tau) = \Bigl(\prod_{i=1}^{n} z_i t_i\Bigr)\Bigl(\prod_{i=1}^{k} z_{\pi(i)} z_{n+\tau(i)}\Bigr) x \Bigl(\prod_{i=k+1}^{\ell} z_{\pi(i)} z_{n+\tau(i)}\Bigr) x \cdot \Bigl(\prod_{i=\ell+1}^{n} z_{\pi(i)} z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_i z_i\Bigr).$$

We note that $\mathbf{w}_n^{0,n}(\pi,\tau) = \mathbf{w}_n(\pi,\tau)$ and $\mathbf{w}_n^{0,0}(\pi,\tau) = \mathbf{w}'_n(\pi,\tau)$.

Lemma 4.10. *Let $\mathbf{X}$ be a monoid variety such that $\mathbf{L} \subseteq \mathbf{X} \subseteq \mathbf{O}$, n be a natural number and $\pi, \tau \in S_n$. If $S(\mathbf{w}_n(\pi,\tau)) \notin \mathbf{X}$ then $\mathbf{X}$ satisfies a non-trivial identity of the form*

$$\mathbf{w}_n(\pi,\tau) \approx \mathbf{w}_n^{k,\ell}(\pi,\tau) \tag{4.19}$$

for some $0 \le k \le \ell \le n$.

Proof. Suppose that $S(\mathbf{w}_n(\pi,\tau)) \notin \mathbf{X}$. Then Lemma 2.3 shows that the variety $\mathbf{X}$ satisfies a non-trivial identity of the form

$$\mathbf{w}_n(\pi,\tau) = \Bigl(\prod_{i=1}^{n} z_i t_i\Bigr) x \Bigl(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\Bigr) x \Bigl(\prod_{i=n+1}^{2n} t_i z_i\Bigr) \approx \mathbf{w}. \tag{4.20}$$

Put $\mathbf{a} = z_{\pi(1)}$, $\mathbf{b} = t_{\pi(1)} z_{\pi(1)+1} t_{\pi(1)+1} \cdots z_n t_n x$, $\mathbf{c} = z_{n+\tau(1)}$ and

$$\mathbf{d} = z_{\pi(2)} z_{n+\tau(2)} \cdots z_{\pi(n)} z_{n+\tau(n)} x t_{n+1} z_{n+1} \cdots t_{n+\tau(1)-1} z_{n+\tau(1)-1} t_{n+\tau(1)}.$$

The word $\mathbf{w}_n(\pi,\tau)$ contains the subword $\mathbf{abacdc}$. Therefore, the submonoid of $S(\mathbf{w}_n(\pi,\tau))$ generated by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ is isomorphic to $S(xzxyty)$. Now Lemmas 2.3 and 4.6 imply that $xzxyty$ is an isoterm for $\mathbf{X}$. Now we are going to verify that

$$\ell_2(\mathbf{w}, z_i) < \ell_1(\mathbf{w}, z_{n+j}) \quad \text{if and only if} \quad \ell_2(\mathbf{w}_n(\pi,\tau), z_i) < \ell_1(\mathbf{w}_n(\pi,\tau), z_{n+j}) \tag{4.21}$$

for any $1 \le i, j \le n$. Indeed, let $1 \le i, j \le n$. The word xyx is an isoterm for $\mathbf{X}$. Since $[\mathbf{w}_n(\pi,\tau)](z_i, t_i) = z_i t_i z_i$ and $[\mathbf{w}_n(\pi,\tau)](z_{n+j}, t_{n+j}) = z_{n+j} t_{n+j} z_{n+j}$, and (4.20) holds in $\mathbf{X}$, we have $\mathbf{w}(z_i,t_i) = z_it_iz_i$ and $\mathbf{w}(z_{n+j},t_{n+j}) = z_{n+j}t_{n+j}z_{n+j}$. The variety $\mathbf{X}$ is non-commutative, whence $\ell_1(\mathbf{w},t_i) < \ell_1(\mathbf{w},t_{n+j})$. Therefore,

$$\mathbf{w}(z_i,t_{n+j}) = [\mathbf{w}_n(\pi,\tau)](z_i,t_{n+j}) = z_i^2t_{n+j},$$
$$\mathbf{w}(z_{n+j},t_i) = [\mathbf{w}_n(\pi,\tau)](z_{n+j},t_i) = t_iz_{n+j}^2.$$

Summarizing, we have

$$\mathbf{w}(z_i,t_i,t_{n+j}) = [\mathbf{w}_n(\pi,\tau)](z_i,t_i,t_{n+j}) = z_it_iz_it_{n+j},$$
$$\mathbf{w}(z_{n+j},t_i,t_{n+j}) = [\mathbf{w}_n(\pi,\tau)](z_{n+j},t_i,t_{n+j}) = t_iz_{n+j}t_{n+j}z_{n+j}.$$

Suppose that $\ell_2(\mathbf{w},z_i) < \ell_1(\mathbf{w},z_{n+j})$. Then the observations in the previous paragraph imply that $\mathbf{w}(z_i,z_{n+j},t_i,t_{n+j}) = z_it_iz_iz_{n+j}t_{n+j}z_{n+j}$. Since $xzxyty$ is an isoterm for $\mathbf{X}$, we have

$$[\mathbf{w}_n(\pi,\tau)](z_i,z_{n+j},t_i,t_{n+j}) = z_it_iz_iz_{n+j}t_{n+j}z_{n+j} = \mathbf{w}(z_i,z_{n+j},t_i,t_{n+j}),$$

whence $\ell_2(\mathbf{w}_n(\pi,\tau),z_i) < \ell_1(\mathbf{w}_n(\pi,\tau),z_{n+j})$.

Suppose now that $\ell_2(\mathbf{w}_n(\pi,\tau),z_i) < \ell_1(\mathbf{w}_n(\pi,\tau),z_{n+j})$. Then

$$[\mathbf{w}_n(\pi,\tau)](z_i,z_{n+j},t_i,t_{n+j}) = z_it_iz_iz_{n+j}t_{n+j}z_{n+j}.$$

Now we apply the fact that $xzxyty$ is an isoterm for $\mathbf{X}$ again to obtain

$$\mathbf{w}(z_i,z_{n+j},t_i,t_{n+j}) = [\mathbf{w}_n(\pi,\tau)](z_i,z_{n+j},t_i,t_{n+j}) = z_it_iz_iz_{n+j}t_{n+j}z_{n+j},$$

whence $\ell_2(\mathbf{w},z_i) < \ell_1(\mathbf{w},z_{n+j})$.

Thus (4.21) is proved. Then

$$\mathbf{w}_x = \Big(\prod_{i=1}^{n} z_it_i\Big)\Big(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Big)\Big(\prod_{i=n+1}^{2n} t_iz_i\Big).$$

Being a subvariety of $\mathbf{O}$, the variety $\mathbf{X}$ satisfies the identities $xyxzx \approx x^2yz \approx yzx^2$. Therefore, we can assume that $\mathrm{occ}_x(\mathbf{w}) = 2$ for any $x \in \mathrm{con}(\mathbf{w})$. So,

$$\mathbf{w} = \Big(\prod_{i=1}^{n} \mathbf{p}_{2i-1}z_i\mathbf{p}_{2i}t_i\Big)\mathbf{q}_0\Big(\prod_{i=1}^{n} z_{\pi(i)}\mathbf{q}_{2i-1}z_{n+\tau(i)}\mathbf{q}_{2i}\Big)\Big(\prod_{i=n+1}^{2n} t_i\mathbf{r}_{2i-2n-1}z_i\mathbf{r}_{2i-2n}\Big)$$

where

$$\Big(\prod_{i=1}^{2n}\mathbf{p}_i\Big)\Big(\prod_{i=0}^{2n}\mathbf{q}_i\Big)\Big(\prod_{i=1}^{2n}\mathbf{r}_i\Big) = x^2.$$

Suppose first that $x \in \mathrm{con}(\mathbf{p}_{2j-1}\mathbf{p}_{2j})$ for some $1 \le j \le n$ and j is the least number with this property. If $\mathbf{p}_{2j-1}\mathbf{p}_{2j} = x$ then

$$\Big(\prod_{i=2j+1}^{2n}\mathbf{p}_i\Big)\Big(\prod_{i=0}^{2n}\mathbf{q}_i\Big)\Big(\prod_{i=1}^{2n}\mathbf{r}_i\Big) = x.$$

It can be easily verified directly that substituting 1 for all letters except x and t_j in (4.20) we obtain the identity $t_jx^2 \approx xt_jx$. But this is impossible because xzx is an isoterm for $\mathbf{X}$. Therefore, $\mathbf{p}_{2j-1}\mathbf{p}_{2j} = x^2$, i.e., either $\mathbf{p}_{2j-1} = \mathbf{p}_{2j} = x$ or $\mathbf{p}_{2j-1} = x^2$ or $\mathbf{p}_{2j} = x^2$. If

$\mathbf{p}_{2j-1} = \mathbf{p}_{2j} = x$ then $\mathbf{X}$ satisfies the identities

$$\begin{aligned}\mathbf{w}_n(\pi,\tau) \approx \mathbf{w} = {} & \Bigl(\prod_{i=1}^{j-1} z_it_i\Bigr)xz_jxt_j\Bigl(\prod_{i=j+1}^{n} z_it_i\Bigr)\Bigl(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr)\\ \stackrel{\sigma_1}{\approx} {} & \Bigl(\prod_{i=1}^{j-1} z_it_i\Bigr)z_jx^2t_j\Bigl(\prod_{i=j+1}^{n} z_it_i\Bigr)\Bigl(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr)\\ \stackrel{(4.15)}{\approx} {} & \Bigl(\prod_{i=1}^{n} z_it_i\Bigr)x^2\Bigl(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr)\\ = {} & \mathbf{w}_n'(\pi,\tau) = \mathbf{w}_n^{0,0}(\pi,\tau),\end{aligned}$$

and we are done. If $\mathbf{p}_{2j-1} = x^2$ or $\mathbf{p}_{2j} = x^2$ then we can apply (4.15) and obtain the required conclusion. So, we can assume that $\mathbf{p}_1 \cdots \mathbf{p}_{2n} = \lambda$.

The case when $x \in \mathrm{con}(\mathbf{r}_{2j-1}\mathbf{r}_{2j})$ for some $1 \le j \le n$ can be considered quite analogously to the previous case with the use of the identity σ_2 rather than σ_1.

Finally, let $x \notin \mathrm{con}(\mathbf{p}_1 \cdots \mathbf{p}_{2n})$ and $x \notin \mathrm{con}(\mathbf{r}_1 \cdots \mathbf{r}_{2n})$. Then $\mathbf{q}_0\mathbf{q}_1 \cdots \mathbf{q}_{2n} = x^2$. Note that either $x \notin \mathrm{con}(\mathbf{q}_0)$ or $x \notin \mathrm{con}(\mathbf{q}_{2n})$ because otherwise (4.20) is trivial. Assume without loss of generality that $x \notin \mathrm{con}(\mathbf{q}_0)$, whence $\mathbf{q}_1 \cdots \mathbf{q}_{2n} = x^2$. Let $x \in \mathrm{con}(\mathbf{q}_k)$ and k is the least number with this property. If $\mathbf{q}_k = x^2$ then (4.15) yields the required conclusion. Suppose now that $x \in \mathrm{con}(\mathbf{q}_{\ell+1})$ for some $k \le \ell \le 2n-1$.

Each occurrence of x in $\mathbf{w}$ lies either in a subword like $z_{\pi(i)}xz_{n+\tau(i)}$ or in a subword like $z_{\pi(i)}z_{n+\tau(i)}xz_{\pi(i+1)}z_{n+\tau(i+1)}$. We need to verify that $\mathbf{w}$ is equal in $\mathbf{X}$ to some word which has the same structure as $\mathbf{w}$ but contains only occurrences of x of the second type. If both occurrences of x in $\mathbf{w}$ are of the second type then we are done. Suppose that both are of the first type. Then $\mathbf{X}$ satisfies the identities

$$\begin{aligned}\mathbf{w}_n(\pi,\tau) \approx \mathbf{w} = {} & \Bigl(\prod_{i=1}^{n} z_it_i\Bigr)\Bigl(\prod_{i=1}^{k-1} z_{\pi(i)}z_{n+\tau(i)}\Bigr)z_{\pi(k)}xz_{n+\tau(k)}\Bigl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\\ & \cdot \underline{z_{\pi(\ell+1)}x}\, z_{n+\tau(\ell+1)}\Bigl(\prod_{i=\ell+2}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr)\\ \stackrel{\sigma_2}{\approx} {} & \Bigl(\prod_{i=1}^{n} z_it_i\Bigr)\Bigl(\prod_{i=1}^{k-1} z_{\pi(i)}z_{n+\tau(i)}\Bigr)z_{\pi(k)}\,\underline{xz_{n+\tau(k)}}\Bigl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\\ & \cdot x\Bigl(\prod_{i=\ell+1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr)\\ \stackrel{\sigma_1}{\approx} {} & \Bigl(\prod_{i=1}^{n} z_it_i\Bigr)\Bigl(\prod_{i=1}^{k} z_{\pi(i)}z_{n+\tau(i)}\Bigr)x\Bigl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\Bigr)x\\ & \cdot \Bigl(\prod_{i=\ell+1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_iz_i\Bigr) = \mathbf{w}_n^{k,\ell}(\pi,\tau)\end{aligned}$$

(for the reader's convenience, we have underlined pairs of adjacent letters that are transposed by σ_1 or σ_2). Finally, if two occurrences of x in $\mathbf{w}$ are of different types, then we can use analogous but simpler arguments. If an occurrence of the first type lies in $\mathbf{q}_k$

[in $\mathbf{q}_{\ell+1}$] then it suffices to apply the identity σ_1 [respectively σ_2] only. Thus, in all cases an identity of the form (4.19) holds in $\mathbf{X}$. ■

Lemma 4.11. *Let m be a natural number, $0 \le k < \ell < m$, $q = \ell - k$ and $\pi, \tau \in S_m$. Then there are permutations $\rho, \sigma \in S_q$ such that the identity $\mathbf{w}_q(\rho, \sigma) \approx \mathbf{w}'_q(\rho, \sigma)$ implies the identity $\mathbf{w}_m^{k,\ell}(\pi, \tau) \approx \mathbf{w}_m^{k,k}(\pi, \tau)$.*

Proof. For convenience, we put $\{z_{\pi(k+1)}, z_{\pi(k+2)}, \dots, z_{\pi(\ell)}\} = \{z_{p_1}, \dots, z_{p_q}\}$ and

$$\{z_{m+\tau(k+1)}, z_{m+\tau(k+2)}, \dots, z_{m+\tau(\ell)}\} = \{z_{r_1}, \dots, z_{r_q}\}$$

where $1 \le p_1 < \cdots < p_q \le m < r_1 < \cdots < r_q \le 2m$. The word $\mathbf{w}_m^{k,\ell}(\pi, \tau)$ has the form

$$\mathbf{u}_0 z_{p_1} \mathbf{u}_1 \cdots z_{p_q} \mathbf{u}_q x z_{\pi(k+1)} z_{m+\tau(k+1)} \cdots z_{\pi(\ell)} z_{m+\tau(\ell)} x \mathbf{u}_{q+1} z_{r_1} \cdots \mathbf{u}_{2q} z_{r_q} \mathbf{u}_{2q+1}$$

where

$$\begin{aligned}
\mathbf{u}_0 &= \prod_{i=1}^{p_1-1} z_i t_i,\\
\mathbf{u}_s &= t_{p_s}\Big(\prod_{i=p_s+1}^{p_{s+1}-1} z_i t_i\Big) \quad \text{for all } 1 \le s < q,\\
\mathbf{u}_q &= t_{p_q}\Big(\prod_{i=p_q+1}^{m} z_i t_i\Big)\Big(\prod_{i=1}^{k} z_{\pi(i)} z_{m+\tau(i)}\Big),\\
\mathbf{u}_{q+1} &= \Big(\prod_{i=\ell+1}^{m} z_{\pi(i)} z_{m+\tau(i)}\Big)\Big(\prod_{i=m+1}^{r_1-1} t_i z_i\Big) t_{r_1},\\
\mathbf{u}_{q+1+s} &= \Big(\prod_{i=r_{s-1}+1}^{r_s-1} t_i z_i\Big) t_{r_s} \quad \text{for all } 1 \le s < q,\\
\mathbf{u}_{2q+1} &= \prod_{i=r_q+1}^{2m} t_i z_i.
\end{aligned}$$

We are going to rename all letters except x in $\mathbf{w}_m^{k,\ell}(\pi, \tau)$. First, we rename all letters from

$$\mathrm{con}(\mathbf{w}_m^{k,\ell}(\pi, \tau)) \setminus \{x, z_{p_1}, \dots, z_{p_q}, z_{r_1}, \dots, z_{r_q}\}$$

to some pairwise different letters that do not occur in $\mathbf{w}_m^{k,\ell}(\pi, \tau)$. Further, we perform the substitution

$$(z_{p_1}, \dots, z_{p_q}, z_{r_1}, \dots, z_{r_q}) \mapsto (z_1, \dots, z_q, z_{q+1}, \dots, z_{2q}).$$

As a result, we get the word

$$\mathbf{u}' = \mathbf{u}'_0 z_1 \mathbf{u}'_1 \cdots z_q \mathbf{u}'_q x z_{\rho(1)} z_{q+\sigma(1)} \cdots z_{\rho(q)} z_{q+\sigma(q)} x \mathbf{u}'_{q+1} z_{q+1} \cdots \mathbf{u}'_{2q} z_{2q} \mathbf{u}'_{2q+1}$$

for some $\rho, \sigma \in S_q$ and some words $\mathbf{u}'_0, \mathbf{u}'_1, \dots, \mathbf{u}'_{2q+1}$.

Now we can perform the substitution $(t_1, \dots, t_{2q}) \mapsto (\mathbf{u}'_1, \dots, \mathbf{u}'_{2q})$ in the identity $\mathbf{w}_q(\rho, \sigma) \approx \mathbf{w}'_q(\rho, \sigma)$. We get the identity

$$z_1\mathbf{u}_1'\cdots z_q\mathbf{u}_q'xz_{\rho(1)}z_{q+\sigma(1)}\cdots z_{\rho(q)}z_{q+\sigma(q)}x\mathbf{u}_{q+1}'z_{q+1}\cdots \mathbf{u}_{2q}'z_{2q} \approx z_1\mathbf{u}_1'\cdots z_q\mathbf{u}_q'x^2z_{\rho(1)}z_{q+\sigma(1)}\cdots z_{\rho(q)}z_{q+\sigma(q)}\mathbf{u}_{q+1}'z_{q+1}\cdots \mathbf{u}_{2q}'z_{2q}.$$

We apply this identity to the word $\mathbf{u}'$ and obtain the identity

$$\mathbf{u}'\approx \mathbf{u}_0'z_1\mathbf{u}_1'\cdots z_q\mathbf{u}_q'x^2z_{\rho(1)}z_{q+\sigma(1)}\cdots z_{\rho(q)}z_{q+\sigma(q)}\mathbf{u}_{q+1}'z_{q+1}\cdots \mathbf{u}_{2q}'z_{2q}\mathbf{u}_{2q+1}'.$$

Now we implement the reverse renaming of letters (to the one described above). We obtain the identity

$$\begin{aligned}\mathbf{w}_m^{k,\ell}(\pi,\tau)&\approx \mathbf{u}_0z_{p_1}\mathbf{u}_1\cdots z_{p_q}\mathbf{u}_qx^2z_{\pi(k+1)}z_{m+\tau(k+1)}\cdots z_{\pi(\ell)}z_{m+\tau(\ell)}x\\&\quad\cdot \mathbf{u}_{q+1}z_{r_1}\cdots \mathbf{u}_{2q}z_{r_q}\mathbf{u}_{2q+1}\\&=\mathbf{w}_m^{k,k}(\pi,\tau).\end{aligned}$$

The lemma is proved. ∎

Now we are well prepared to complete the proof of necessity of Theorem 1.1. Recall that we have reduced our considerations to the case $\mathbf{L}\subseteq\mathbf{V}\subseteq\mathbf{O}$. We denote by $\mathcal{K}$ the class of all varieties of the form $\operatorname{var} S(\mathbf{w}_n(\pi,\tau))$ where $n\in\mathbb{N}$ and $\pi,\tau\in S_n$. It is clear that $\mathbf{L}\subseteq\mathbf{X}$ whenever $\mathbf{X}\in\mathcal{K}$. We use this fact below without explicit mention. Let $\mathbf{X}\in\mathcal{K}$. We are going to verify that $\mathbf{X}$ contains at least two incomparable subvarieties from $\mathcal{K}$.

For any $\xi\in S_n$, we define the following two permutations from S_{n+2}:

$$\xi_1=\begin{pmatrix}1&2&3&4&5&\dots&n+2\\ \xi(1)+2&1&2&\xi(2)+2&\xi(3)+2&\dots&\xi(n)+2\end{pmatrix},$$

$$\xi_2=\begin{pmatrix}1&2&3&4&5&\dots&n+2\\ \xi(1)+2&2&1&\xi(2)+2&\xi(3)+2&\dots&\xi(n)+2\end{pmatrix}.$$

We have $\mathbf{X}=\operatorname{var} S(\mathbf{w}_n(\pi,\tau))$ for some n, π and τ. Let $T_1=S_{n+2}(\pi_1,\tau_1)$ and $T_2=S_{n+2}(\pi_2,\tau_1)$. If $T_1\notin\mathbf{X}$ then Lemma 4.10 allows us to assume that $\mathbf{X}$ satisfies a non-trivial identity of the form $\mathbf{w}_{n+2}(\pi_1,\tau_1)\approx\mathbf{w}_{n+2}^{k,\ell}(\pi_1,\tau_1)$ for some $1\le k\le\ell\le n+2$. Then we substitute

- 1 for z_1, z_2, z_{n+3}, z_{n+4}, t_1, t_2, t_{n+3} and t_{n+4},
- z_{i-2} for z_i whenever $3\le i\le n+2$ and z_{i-4} for z_i whenever $n+5\le i\le 2n+4$,
- t_{i-2} for t_i whenever $3\le i\le n+2$ and t_{i-4} for t_i whenever $n+5\le i\le 2n+4$.

Thus $\mathbf{X}$ satisfies the identity $\mathbf{w}_n(\pi,\tau)\approx\mathbf{w}_n^{s,t}(\pi,\tau)$ where $s=1$ whenever $k\le 3$ and $s=k-2$ whenever $k>3$, while $t=1$ whenever $\ell\le 3$ and $t=\ell-2$ whenever $\ell>3$. Since $s\ge 1$, this identity is non-trivial. This contradicts the fact that $\mathbf{X}=\operatorname{var} S(\mathbf{w}_n(\pi,\tau))$ and Lemma 2.3. Thus, we have proved that $T_1\in\mathbf{X}$. Analogously, $T_2\in\mathbf{X}$.

Suppose that $T_1\in\operatorname{var} T_2$. Then Lemma 2.3 shows that $\mathbf{w}_{n+2}(\pi_1,\tau_1)$ is an isoterm for $\operatorname{var} T_2$. At the same time, it is easy to verify that $\operatorname{var} T_2$ satisfies $\mathbf{w}_{n+2}(\pi_1,\tau_1)\approx\mathbf{w}_{n+2}'(\pi_1,\tau_1)$. Therefore, $\operatorname{var} T_1\nsubseteq\operatorname{var} T_2$. Analogously, $\operatorname{var} T_2\nsubseteq\operatorname{var} T_1$. We see that $\operatorname{var} T_1$ and $\operatorname{var} T_2$ are incomparable. Moreover, it is evident that these two varieties lie in $\mathcal{K}$.

Thus, if $\mathbf{X}=\operatorname{var} S(\mathbf{w}_n(\pi,\tau))$ for some n, π and τ then $\mathbf{X}$ is not a chain variety. Therefore, $S(\mathbf{w}_n(\pi,\tau))\notin\mathbf{V}$ for all n, π and τ. For any n, we denote by ε the trivial permutation from S_n. Then $S(\mathbf{w}_1(\varepsilon,\varepsilon))\notin\mathbf{V}$. According to Lemma 4.10, $\mathbf{V}$ satisfies a non-trivial identity $\mathbf{w}_1(\varepsilon,\varepsilon)\approx\mathbf{w}_1^{k,\ell}(\varepsilon,\varepsilon)$ with $0\le k\le\ell\le 1$. Since $\mathbf{w}_1^{0,0}(\varepsilon,\varepsilon)=$

$\mathbf{w}_1'(\varepsilon,\varepsilon)$, $\mathbf{w}_1^{0,1}(\varepsilon,\varepsilon)=\mathbf{w}_1(\varepsilon,\varepsilon)$ and the identity $\mathbf{w}_1(\varepsilon,\varepsilon)\approx\mathbf{w}_1^{k,\ell}(\varepsilon,\varepsilon)$ is non-trivial, $\mathbf{V}$ satisfies $\mathbf{w}_1(\varepsilon,\varepsilon)\approx\mathbf{w}_1'(\varepsilon,\varepsilon)$ or $\mathbf{w}_1(\varepsilon,\varepsilon)\approx\mathbf{w}_1^{1,1}(\varepsilon,\varepsilon)$. Clearly, the latter identity together with (4.15) implies the former. Thus, $\mathbf{V}$ satisfies $\mathbf{w}_1(\varepsilon,\varepsilon)\approx\mathbf{w}_1'(\varepsilon,\varepsilon)$.

Thus, there is a number n such that $\mathbf{V}$ satisfies the identities (4.12) for all $\pi,\tau\in S_n$ (for instance, $n=1$). We are going to verify that every n has this property. Towards a contradiction, suppose that the above claim is true for $1,\dots,n$ but false for $n+1$. Let $\pi_1,\tau_1\in S_{n+1}$. Since $S(\mathbf{w}_{n+1}(\pi_1,\tau_1))\notin\mathbf{V}$, Lemma 4.10 implies that $\mathbf{V}$ satisfies $\mathbf{w}_{n+1}(\pi_1,\tau_1)\approx\mathbf{w}_{n+1}^{k,\ell}(\pi_1,\tau_1)$ for some $0\le k\le\ell<n+1$. Suppose that $k<\ell$. Then Lemma 4.11 with $m=n+1$, $\pi=\pi_1$ and $\tau=\tau_1$ shows that there exist $\rho,\sigma\in S_{\ell-k}$ such that the identity $\mathbf{w}_{\ell-k}(\rho,\sigma)\approx\mathbf{w}_{\ell-k}'(\rho,\sigma)$ implies $\mathbf{w}_{n+1}^{k,\ell}(\pi_1,\tau_1)\approx\mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1)$. The former identity holds in $\mathbf{V}$ because $\ell-k\le n$. Thus, in any case $\mathbf{V}$ satisfies $\mathbf{w}_{n+1}(\pi_1,\tau_1)\approx\mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1)$. Note that $\mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1)\stackrel{(4.15)}{\approx}\mathbf{w}_{n+1}^{0,0}(\pi_1,\tau_1)=\mathbf{w}_{n+1}'(\pi_1,\tau_1)$. Therefore, $\mathbf{w}_{n+1}(\pi_1,\tau_1)\approx\mathbf{w}_{n+1}'(\pi_1,\tau_1)$ holds in $\mathbf{V}$ for any $\pi_1,\tau_1\in S_{n+1}$. This contradicts the choice of n. So, the variety $\mathbf{V}$ satisfies the identities (4.12) for all n and $\pi,\tau\in S_n$, whence $\mathbf{V}=\mathbf{L}$.

We have thus completed the proof of the "only if" part of Theorem 1.1.

5. The proof of the "if" part: all varieties except K

In this and the following chapters we are going to prove that if $\mathbf{X}$ is a subvariety of one of the varieties listed in Theorem 1.1 then $\mathbf{X}$ is a chain variety. Since the property of being a chain variety is inherited by subvarieties, we can assume that $\mathbf{X}$ coincides with one of the varieties listed in Theorem 1.1. By symmetry, we can exclude the varieties $\overleftarrow{\mathbf{K}}$ and $\overleftarrow{\mathbf{N}}$. Thus, it suffices to verify that $\mathbf{C}_n$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{N}$ and $\mathbf{RRB}$ are chain varieties. Here we consider all these varieties except $\mathbf{K}$, which will be examined in the next chapter.

Lemmas 2.8 and 2.9(i) and Corollary 4.7 immediately imply that $\mathbf{D}$, $\mathbf{L}$, $\mathbf{LRB}$ and $\mathbf{RRB}$ are chain varieties.

Proposition 5.1. *The lattice $L(\mathbf{C}_n)$ is the chain*

$$\mathbf{T}\subset\mathbf{SL}\subset\mathbf{C}_2\subset\cdots\subset\mathbf{C}_n.$$

Proof. Let $\mathbf{V}\subseteq\mathbf{C}_n$. Then $\mathbf{V}$ is commutative and aperiodic. If $\mathbf{C}_2\nsubseteq\mathbf{V}$ then $\mathbf{V}$ is completely regular by Corollary 2.6. Then $\mathbf{V}\subseteq\mathbf{SL}$, whence $\mathbf{V}$ coincides with either $\mathbf{T}$ or $\mathbf{SL}$. It remains to verify that if $\mathbf{C}_2\subseteq\mathbf{V}\subseteq\mathbf{C}_n$ then $\mathbf{V}=\mathbf{C}_s$ for some $2\le s\le n$. We will use induction on n. If $n=2$ then the assertion is obvious. Let now $n>2$. Suppose that $\mathbf{V}\ne\mathbf{C}_n$. Then Lemma 2.5 implies that $\mathbf{V}$ satisfies the identity $x^{n-1}\approx x^n$, whence $\mathbf{V}\subseteq\mathbf{C}_{n-1}$. By the induction assumption, $\mathbf{V}=\mathbf{C}_s$ for some $2\le s\le n-1$. ∎

It remains to consider the variety $\mathbf{N}$.

Proposition 5.2. *The lattice $L(\mathbf{N})$ is the chain*

$$\mathbf{T}\subset\mathbf{SL}\subset\mathbf{C}_2\subset\mathbf{D}_1\subset\mathbf{D}_2\subset\mathbf{M}\subset\mathbf{N}.$$

Proof. First of all, we are going to check that $\mathbf{N}$ satisfies identities of the form (4.11) for all n, m and $\theta\in S_{n+m}$. Indeed, $\mathbf{w}_{n,m}(\theta)=\mathbf{p}x\mathbf{q}x\mathbf{r}$ where $\mathbf{p}=z_1t_1\cdots z_nt_n$, $\mathbf{q}=$

$z_{\theta(1)}\cdots z_{\theta(n+m)}$ and $\mathbf{r}=t_{n+1}z_{n+1}\cdots t_{n+m}z_{n+m}$. Suppose first that $\theta(n+m)\le n$. Then

$$\mathbf{w}_{n,m}(\theta)=z_1t_1\cdots \overset{(1)}{z_{\theta(n+m)}}\, t_{\theta(n+m)}\cdots z_nt_n\, \overset{(1)}{x}\, z_{\theta(1)}\cdots \overset{(2)}{z_{\theta(n+m)}}\,\overset{(2)}{x}\,\mathbf{r}.$$

We see that the second occurrences of the letters $z_{\theta(n+m)}$ and x in $\mathbf{w}_{n,m}(\theta)$ are adjacent to each other. The identity σ_2 allows us to swap these occurrences. In other words,

$$\mathbf{w}_{n,m}(\theta)\stackrel{\sigma_2}{\approx}\mathbf{p}xz_{\theta(1)}\cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}.$$

Suppose now that $\theta(n+m)>n$. Then

$$\mathbf{w}_{n,m}(\theta)=\mathbf{p}\,\overset{(1)}{x}\,z_{\theta(1)}\cdots \overset{(1)}{z_{\theta(n+m)}}\,\overset{(2)}{x}\,t_{n+1}z_{n+1}\cdots t_{\theta(n+m)}\,\overset{(2)}{z_{\theta(n+m)}}\cdots t_{n+m}z_{n+m}.$$

We see that the first occurrence of $z_{\theta(n+m)}$ and the second occurrence x in $\mathbf{w}_{n,m}(\theta)$ are adjacent to each other. The identity γ_1 allows us to transpose these occurrences. In other words,

$$\mathbf{w}_{n,m}(\theta)\stackrel{\gamma_1}{\approx}\mathbf{p}xz_{\theta(1)}\cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}.$$

We see that in any case the identity

$$\mathbf{w}_{n,m}(\theta)\approx\mathbf{p}xz_{\theta(1)}\cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}$$

holds in $\mathbf{N}$. Analogous arguments show that we can successively swap the second occurrence of x with $z_{\theta(n+m-1)}$, $z_{\theta(n+m-2)}$, …, $z_{\theta(1)}$ and deduce that $\mathbf{N}$ satisfies the identities

$$\mathbf{w}_{n,m}(\theta)\approx\mathbf{p}x^2z_{\theta(1)}\cdots z_{\theta(n+m)}\mathbf{r}=\mathbf{p}x^2\mathbf{q}\mathbf{r}=\mathbf{w}'_{n,m}(\theta).$$

Therefore, we can apply Lemma 4.5 below.

Suppose that $\mathbf{V}\subseteq\mathbf{N}$. If $\mathbf{M}\nsubseteq\mathbf{V}$ then $\mathbf{V}\subseteq\mathbf{D}$ by Lemma 4.9(ii). Therefore, in view of Lemma 2.8, it suffices to consider the case when $\mathbf{M}\subseteq\mathbf{V}$. We need to verify that $\mathbf{V}$ coincides with $\mathbf{M}$ or $\mathbf{N}$. Let $\mathbf{u}\approx\mathbf{v}$ be an identity that holds in $\mathbf{V}$. Our aim is to verify that $\mathbf{u}\approx\mathbf{v}$ either implies the identity α_1 or holds in $\mathbf{N}$. Proposition 2.2 implies that $\mathrm{sim}(\mathbf{u})=\mathrm{sim}(\mathbf{v})$. Let $\mathrm{sim}(\mathbf{u})=\{t_0,t_1,\dots,t_m\}$. As in the proof of Lemma 4.6, we can assume that

$$\mathbf{u}=t_0\mathbf{a}_1t_1\mathbf{a}_2t_2\cdots t_{m-1}\mathbf{a}_mt_m\quad\text{and}\quad\mathbf{v}=t_0\mathbf{b}_1t_1\mathbf{b}_2t_2\cdots t_{m-1}\mathbf{b}_mt_m$$

for some possibly empty words $\mathbf{a}_1,\dots,\mathbf{a}_m$ and $\mathbf{b}_1,\dots,\mathbf{b}_m$.

Let x be a letter multiple in $\mathbf{u}$ and $\mathbf{u}(x,y)\ne xyx$ for any letter y. By Lemma 4.5, the variety $\mathbf{V}$ satisfies (4.16). Obviously, $\mathbf{C}_2\subseteq\mathbf{M}\subseteq\mathbf{V}$, whence $x\in\mathrm{mul}(\mathbf{v})$ by Proposition 2.2. Since $\mathbf{D}_2\subseteq\mathbf{M}\subseteq\mathbf{V}$, we apply Lemmas 2.3 and 2.7 to conclude that xyx is an isoterm for $\mathbf{V}$. Therefore, $\mathbf{v}(x,y)\ne xyx$ for any letter y. We apply Lemma 4.5 again to conclude that $\mathbf{v}\approx x^2\mathbf{v}_x$ holds in $\mathbf{V}$. Thus, $\mathbf{u}\approx\mathbf{v}$ follows from the identities (4.16), $\mathbf{v}\approx x^2\mathbf{v}_x$ and $\mathbf{u}_x\approx\mathbf{v}_x$. So, we can remove from $\mathbf{u}\approx\mathbf{v}$ all multiple letters x such that $\mathbf{u}(x,y)\ne xyx$ for any y. In other words, we may assume that for any $x\in\mathrm{mul}(\mathbf{u})$ there is a letter y such that $\mathbf{u}(x,y)=xyx=\mathbf{v}(x,y)$. In particular, $\mathrm{occ}_x(\mathbf{u}),\mathrm{occ}_x(\mathbf{v})\le 2$ for any letter x.

Let $0 \le i \le m-1$. Then $\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2$ where

$$\mathbf{w}_1 = \begin{cases} t_0 \mathbf{a}_1 t_1 \cdots t_{i-1} \mathbf{a}_i & \text{if } 0 < i \le m-1, \\ \lambda & \text{if } i = 0, \end{cases}$$

$$\mathbf{w}_2 = \begin{cases} \mathbf{a}_{i+2} t_{i+2} \cdots \mathbf{a}_m t_m & \text{if } 0 \le i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

Analogously, $\mathbf{v} = \mathbf{w}_1' t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2'$ where

$$\mathbf{w}_1' = \begin{cases} t_0 \mathbf{b}_1 t_1 \cdots t_{i-1} \mathbf{b}_i & \text{if } 0 < i \le m-1, \\ \lambda & \text{if } i = 0, \end{cases}$$

$$\mathbf{w}_2' = \begin{cases} \mathbf{b}_{i+2} t_{i+2} \cdots \mathbf{b}_m t_m & \text{if } 0 \le i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

Suppose that $\mathbf{a}_{i+1}$ contains the subword $\mathbf{d} = x_i x_j$ where $x_i \in \mathrm{con}(\mathbf{w}_1)$ and $x_j \in \mathrm{con}(\mathbf{w}_2)$. The occurrence of the letter x_i in $\mathbf{d}$ is the second occurrence of x_i in $\mathbf{u}$, while the occurrence of x_j in $\mathbf{d}$ is the first occurrence of x_j in $\mathbf{u}$. The identity γ_1 allows us to swap these two occurrences. Therefore, $\mathbf{N}$ satisfies the identity $\mathbf{u} \approx \mathbf{w}_1 t_i \mathbf{p}_1 \mathbf{q}_1 t_{i+1} \mathbf{w}_2$ where $\mathrm{con}(\mathbf{p}_1) \subseteq \mathrm{con}(\mathbf{w}_2)$ and $\mathrm{con}(\mathbf{q}_1) \subseteq \mathrm{con}(\mathbf{w}_1)$. Analogously, we can prove that $\mathbf{N}$ satisfies $\mathbf{v} \approx \mathbf{w}_1' t_i \mathbf{p}_2 \mathbf{q}_2 t_{i+1} \mathbf{w}_2'$ where $\mathrm{con}(\mathbf{p}_2) \subseteq \mathrm{con}(\mathbf{w}_2')$ and $\mathrm{con}(\mathbf{q}_2) \subseteq \mathrm{con}(\mathbf{w}_1')$.

We are going to verify that $\mathrm{con}(\mathbf{p}_1) = \mathrm{con}(\mathbf{p}_2)$ and $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2)$. Let $x \in \mathrm{con}(\mathbf{p}_1)$. Then $\mathbf{u}(x, t_{i+1}) = x t_{i+1} x$. Therefore, $\mathbf{v}(x, t_{i+1}) = x t_{i+1} x$. This means that $x \in \mathrm{con}(\mathbf{w}_1' \mathbf{p}_2 \mathbf{q}_2)$ and $x \in \mathrm{con}(\mathbf{w}_2')$. If $x \in \mathrm{con}(\mathbf{q}_2)$ then $x \in \mathrm{con}(\mathbf{w}_1')$ as well, whence $\mathrm{occ}_x(\mathbf{v}) \ge 3$. Therefore, $x \notin \mathrm{con}(\mathbf{q}_2)$. Note that $\mathbf{u}(x, t_i) = t_i x^2$. Therefore, $\mathbf{v}(x, t_i) \ne x t_i x$, whence $x \notin \mathrm{con}(\mathbf{w}_1')$. We see that $x \in \mathrm{con}(\mathbf{p}_2)$. We have just proved that $\mathrm{con}(\mathbf{p}_1) \subseteq \mathrm{con}(\mathbf{p}_2)$. By symmetry, $\mathrm{con}(\mathbf{p}_2) \subseteq \mathrm{con}(\mathbf{p}_1)$, whence $\mathrm{con}(\mathbf{p}_1) = \mathrm{con}(\mathbf{p}_2)$. Analogous arguments imply that $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2)$.

Therefore, $\mathbf{p}_1 = x_1 \cdots x_k$ and $\mathbf{p}_2 = x_{\pi(1)} \cdots x_{\pi(k)}$ for some $x_1, \ldots, x_k \in \mathrm{con}(\mathbf{w}_2) \cap \mathrm{con}(\mathbf{w}_2')$ and some $\pi \in S_k$, whence $\mathbf{N}$ satisfies

$$\mathbf{u} \approx \mathbf{w}_1 t_i x_1 \cdots x_k \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \quad \text{and} \quad \mathbf{v} \approx \mathbf{w}_1' t_i x_{\pi(1)} \cdots x_{\pi(k)} \mathbf{q}_2 t_{i+1} \mathbf{w}_2'.$$

Then the identity

$$\mathbf{w}_1 t_i x_1 \cdots x_k \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1' t_i x_{\pi(1)} \cdots x_{\pi(k)} t_{i+1} \mathbf{q}_2 \mathbf{w}_2' \tag{5.1}$$

holds in $\mathbf{V}$.

Suppose that π is non-trivial. Then there are j and ℓ such that $j < \ell$ but $\pi(j) > \pi(\ell)$. Substituting 1 for all letters occurring in (5.1) except x_j, x_ℓ and t_{i+1}, we obtain $x_j x_\ell t_{i+1} \mathbf{s} \approx x_\ell x_j t_{i+1} \mathbf{s}'$ where $\mathbf{s}, \mathbf{s}' \in \{x_j x_\ell, x_\ell x_j\}$. Now we apply σ_2 to get $x_j x_\ell t_{i+1} x_j x_\ell \approx x_\ell x_j t_{i+1} x_j x_\ell$. The last identity is nothing but α_1 (up to renaming the letters). So, if π is non-trivial then $\mathbf{V}$ satisfies α_1. This means that $\mathbf{V} \subseteq \mathbf{M}$, whence $\mathbf{V} = \mathbf{M}$. In other words, if $\mathbf{p}_1 \ne \mathbf{p}_2$ then $\mathbf{V} = \mathbf{M}$.

Let now $\mathbf{p}_1 = \mathbf{p}_2$. The words $\mathbf{q}_1$ and $\mathbf{q}_2$ are linear and $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2) \subseteq \mathrm{con}(\mathbf{w}_1) \cap \mathrm{con}(\mathbf{w}_1')$. Thus, if some letter z occurs in $\mathrm{con}(\mathbf{q}_1)$ then this occurrence is the second occurrence of z in $\mathbf{u}$. Hence the identity σ_2 allows us to reorder the letters in $\mathbf{q}_1$ in an arbitrary way. Therefore, we can replace $\mathbf{q}_1$ by $\mathbf{q}_2$ in $\mathbf{u}$, and the resulting word should be equal to $\mathbf{u}$ in $\mathbf{N}$. Thus, $\mathbf{N}$ satisfies the identities

$$\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{p}_1 \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{p}_2 \mathbf{q}_2 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2.$$

This is true for all $i = 0, 1, \ldots, m-1$. Therefore, $\mathbf{N}$ satisfies

$$\mathbf{u} = t_0 \mathbf{a}_1 t_1 \mathbf{a}_2 t_2 \cdots t_{m-1} \mathbf{a}_m t_m \approx t_0 \mathbf{b}_1 t_1 \mathbf{b}_2 t_2 \cdots t_{m-1} \mathbf{b}_m t_m = \mathbf{v}.$$

The proposition is proved. ■

6. The proof of the "if" part: the variety K

Here we are going to verify that $\mathbf{K}$ is a chain variety. This case is much more complex than all those discussed in the previous chapter, and its consideration will be much longer. For the reader's convenience, we divide this chapter into four sections.

6.1. Reduction to the interval $[\mathbf{E}, \mathbf{K}]$. We fix notation for the following identity system:

$$\Phi = \{xyx \approx xyx^2,\ x^2y^2 \approx y^2x^2,\ x^2y \approx x^2yx\}.$$

Note that $\mathbf{K} = \operatorname{var}\Phi$. For any $s \in \mathbb{N}$ and $1 \le q \le s$, we put

$$\mathbf{b}_{s,q} = x_{s-1}x_s x_{s-2} x_{s-1} \cdots x_{q-1} x_q.$$

For brevity, we will write $\mathbf{b}_s$ rather than $\mathbf{b}_{s,1}$. We also put $\mathbf{b}_0 = \lambda$ for convenience. We introduce the following four countably infinite series of identities:

$$\begin{aligned}
\alpha_k &: \ x_k y_k x_{k-1} x_k y_k \mathbf{b}_{k-1} \approx y_k x_k x_{k-1} x_k y_k \mathbf{b}_{k-1},\\
\beta_k &: \ x x_k x \mathbf{b}_k \approx x_k x^2 \mathbf{b}_k,\\
\gamma_k &: \ y_1 y_0 x_k y_1 \mathbf{b}_k \approx y_1 y_0 y_1 x_k \mathbf{b}_k,\\
\delta_k^m &: \ y_{m+1} y_m x_k y_{m+1} \mathbf{b}_{k,m} y_m \mathbf{b}_{m-1} \approx y_{m+1} y_m y_{m+1} x_k \mathbf{b}_{k,m} y_m \mathbf{b}_{m-1}
\end{aligned}$$

where $k \in \mathbb{N}$ and $1 \le m \le k$. Note that the identities α_1 and γ_1 have already appeared above. We define the following four countably infinite series of varieties:

$$\mathbf{F}_k = \operatorname{var}\{\Phi, \alpha_k\}, \quad \mathbf{H}_k = \operatorname{var}\{\Phi, \beta_k\}, \quad \mathbf{I}_k = \operatorname{var}\{\Phi, \gamma_k\}, \quad \mathbf{J}_k^m = \operatorname{var}\{\Phi, \delta_k^m\}.$$

In this chapter we are going to verify

Proposition 6.1.

(1) *The lattice $L(\mathbf{K})$ is the set-theoretical union of the lattice $L(\mathbf{E})$ and the interval $[\mathbf{E}, \mathbf{K}]$.*
(2) *The lattice $L(\mathbf{E})$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{E}$.*
(3) *If $\mathbf{X}$ is a monoid variety such that $\mathbf{E} \subset \mathbf{X} \subset \mathbf{K}$ then $\mathbf{X}$ belongs to the interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ for some k.*
(4) *The interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ is the chain*

$$\mathbf{F}_k \subset \mathbf{H}_k \subset \mathbf{I}_k \subset \mathbf{J}_k^1 \subset \mathbf{J}_k^2 \subset \cdots \subset \mathbf{J}_k^k \subset \mathbf{F}_{k+1}. \tag{6.1}$$

This proposition immediately implies that $L(\mathbf{K})$ is the following chain:

$$\begin{aligned}\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{E} &\subset \mathbf{F}_1 \subset \mathbf{H}_1 \subset \mathbf{I}_1 \subset \mathbf{J}_1^1\\ &\subset \mathbf{F}_2 \subset \mathbf{H}_2 \subset \mathbf{I}_2 \subset \mathbf{J}_2^1 \subset \mathbf{J}_2^2\\ &\vdots\\ &\subset \mathbf{F}_k \subset \mathbf{H}_k \subset \mathbf{I}_k \subset \mathbf{J}_k^1 \subset \mathbf{J}_k^2 \subset \cdots \subset \mathbf{J}_k^k\\ &\vdots\\ &\subset \mathbf{K}.\end{aligned}$$

In the remainder of this section we verify claim (1) of Proposition 6.1. Claim (2) follows from Lemma 2.10(i). Claims (3) and (4) are proved in Sections 6.3 and 6.4 respectively. Section 6.2 contains auxiliary assertions.

Let $\mathbf{X}$ be a monoid variety with $\mathbf{X} \subseteq \mathbf{K}$. We need to verify that either $\mathbf{E} \subseteq \mathbf{X}$ or $\mathbf{X} \subseteq \mathbf{E}$. Substituting 1 for y in the identity (4.9), we find that $\mathbf{X}$ satisfies the identity (4.5). If $\mathbf{X}$ is commutative then $\mathbf{X} \subseteq \mathbf{C}_2 \subseteq \mathbf{E}$, and we are done. Thus, we can assume that $\mathbf{X}$ is non-commutative. The variety $\mathbf{X}$ is aperiodic because it satisfies (4.5). Suppose that $\mathbf{X}$ is completely regular. Every aperiodic completely regular variety is a variety of band monoids and every band satisfying (4.4) is commutative. Thus, if $\mathbf{X}$ is completely regular then it is commutative, a contradiction. Hence we can assume that $\mathbf{X}$ is not completely regular. Then $\mathbf{D}_1 \subseteq \mathbf{X}$ by Lemma 2.14.

Suppose that $\mathbf{E} \nsubseteq \mathbf{X}$. Then $\mathbf{X}$ satisfies (4.6) by Lemma 4.3. Further, $\mathbf{X}$ satisfies (4.10) as well because $\mathbf{X} \subseteq \mathbf{K}$. Hence $x^2y \stackrel{(4.10)}{\approx} x^2yx^2 \stackrel{(4.6)}{\approx} yx^2$. We see that (4.15) holds in $\mathbf{X}$. Moreover,

$$xyx \stackrel{(4.9)}{\approx} xyx^2 \stackrel{(4.6)}{\approx} x^3yx^2 \stackrel{(4.5)}{\approx} x^2yx^2 \stackrel{(4.6)}{\approx} yx^2 \stackrel{(4.15)}{\approx} x^2y,$$

whence the identity

$$xyx \approx x^2y \tag{6.2}$$

holds in $\mathbf{X}$. So, $\mathbf{X}$ satisfies $yx^2 \stackrel{(4.15)}{\approx} x^2y \stackrel{(6.2)}{\approx} xyx$. The identities (4.15), (4.4) and (6.2) evidently imply σ_1, σ_2 and γ_1. Thus, $\mathbf{X} \subseteq \mathbf{D}_1 \subseteq \mathbf{E}$. We have proved that if $\mathbf{E} \nsubseteq \mathbf{X}$ then $\mathbf{X} \subseteq \mathbf{E}$. Hence claim (1) of Proposition 6.1 is proved.

6.2. Several auxiliary results. Here we prove several lemmas that will be used repeatedly below.

6.2.1. Some properties of the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^m$, $\mathbf{K}$ and their identities

LEMMA 6.2. *The variety* $\mathbf{K}$ *satisfies:*

(i) *the identity* σ_2*;*

(ii) *the identity*

$$xyxzx \approx xyxz; \tag{6.3}$$

(iii) *any identity* $\mathbf{u} \approx \mathbf{v}$ *such that* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ *and* $\mathrm{occ}_x(\mathbf{u}), \mathrm{occ}_x(\mathbf{v}) \geq 2$ *for any letter* $x \in \mathrm{con}(\mathbf{u})$.

Proof. (i) We have $xzytxy \stackrel{(4.9)}{\approx} xzytx^2y^2 \stackrel{(4.4)}{\approx} xzyty^2x^2 \stackrel{(4.9)}{\approx} xzytyx$.

(ii) We have $xyxzx \stackrel{(4.9)}{\approx} xyx^2zx \stackrel{(4.10)}{\approx} xyx^2z \stackrel{(4.9)}{\approx} xyxz$.

(iii) According to (ii), $\mathbf{V}$ satisfies (6.3). This allows us to assume that $\mathrm{occ}_x(\mathbf{u}) = \mathrm{occ}_x(\mathbf{v}) = 2$ for any $x \in \mathrm{con}(\mathbf{u})$. Let $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v}) = \{x_1, \dots, x_k\}$. We are going to verify that $\mathbf{u} \approx x_1^2 \cdots x_k^2$ in $\mathbf{K}$. We will use induction on k.

Induction base. Suppose that $k = 1$. Here the identity $\mathbf{u} \approx \mathbf{v}$ has the form $x_1^2 \approx x_1^2$, whence it trivially holds in $\mathbf{K}$.

Induction step. Let now $k > 1$. We may assume that $\ell_1(\mathbf{u}, x_i) < \ell_1(\mathbf{u}, x_k)$ for any $1 \le i < k$. Then

$$\mathbf{u} = \mathbf{u}' x_k x_{j_1} x_{j_2} \cdots x_{j_s} x_k x_{j_{s+1}} x_{j_{s+2}} \cdots x_{j_{s+t}}$$

where $x_{j_r} \in \mathrm{con}(\mathbf{u}')$ for any $1 \le r \le s + t$. Then the identities

$$\begin{aligned}\mathbf{u} &\stackrel{(4.9)}{\approx} \mathbf{u}' x_k x_{j_1}^2 x_{j_2}^2 \cdots x_{j_s}^2 x_k^2 x_{j_{s+1}}^2 x_{j_{s+2}}^2 \cdots x_{j_{s+t}}^2 \\ &\stackrel{(4.4)}{\approx} \mathbf{u}' x_k^3 x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 \\ &\stackrel{(4.5)}{\approx} \mathbf{u}' x_k^2 x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 \\ &\stackrel{(4.4)}{\approx} \mathbf{u}' x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 x_k^2 \\ &\stackrel{(6.3)}{\approx} \mathbf{u}' x_{j_1} x_{j_2} \cdots x_{j_{s+t}} x_k^2 = \mathbf{u}_{x_k} x_k^2\end{aligned}$$

hold in $\mathbf{K}$. The word $\mathbf{u}_{x_k}$ contains exactly $k-1$ letters. By the induction assumption, the identity $\mathbf{u}_{x_k} \approx x_1^2 \cdots x_{k-1}^2$ holds in $\mathbf{K}$, whence this variety satisfies $\mathbf{u} \approx \mathbf{u}_{x_k} x_k^2 \approx x_1^2 \cdots x_k^2$. Similarly, $\mathbf{v} \approx x_1^2 \cdots x_k^2$ in $\mathbf{K}$, whence $\mathbf{K}$ satisfies $\mathbf{u} \approx \mathbf{v}$. ■

LEMMA 6.3. *The identity system Φ together with the identity*

$$x x_k x \mathbf{b}_k \approx x^2 x_k \mathbf{b}_k \tag{6.4}$$

forms an identity basis of the variety $\mathbf{J}_k^k$.

Proof. First of all, we note that (6.4) holds in $\mathbf{J}_k^k$. To check this, it suffices to perform the substitution $(y_k, y_{k+1}) \mapsto (1, x)$ in the identity δ_k^k and use the equality $\mathbf{b}_k = x_{k-1} x_k \mathbf{b}_{k-1}$. So, we need to verify that δ_k^k follows from Φ and (6.4). In view of Lemma 6.2, we can use the identities σ_2 and (6.3). Here is the required deduction (letters in the right column refer to comments after the deduction):

$$\begin{aligned} y_{k+1} y_k x_k y_{k+1} \mathbf{b}_{k,k} y_k \mathbf{b}_{k-1} &= y_{k+1} y_k x_k y_{k+1} x_{k-1} x_k y_k \mathbf{b}_{k-1} && \text{(a)} \\ &\approx y_{k+1} y_k x_k y_{k+1} x_{k-1} y_k x_k \mathbf{b}_{k-1} && \text{(b)} \\ &\approx y_{k+1}^2 y_k x_k x_{k-1} y_k x_k \mathbf{b}_{k-1} && \text{(c)} \\ &\approx y_{k+1}^2 y_k x_k x_{k-1} x_k y_k \mathbf{b}_{k-1} && \text{(d)} \\ &= y_{k+1}^2 y_k x_k x_{k-1} x_k y_k x_{k-2} x_{k-1} \mathbf{b}_{k-2} && \text{(e)} \\ &\approx y_{k+1}^2 y_k x_k x_{k-1} x_k y_k x_{k-2} x_k x_{k-1} x_k \mathbf{b}_{k-2} && \text{(f)} \\ &\approx y_{k+1} y_k y_{k+1} x_k x_{k-1} x_k y_k x_{k-2} x_k x_{k-1} x_k \mathbf{b}_{k-2} && \text{(g)} \\ &\approx y_{k+1} y_k y_{k+1} x_k x_{k-1} x_k y_k x_{k-2} x_{k-1} \mathbf{b}_{k-2} && \text{(h)} \\ &= y_{k+1} y_k y_{k+1} x_k \mathbf{b}_{k,k} y_k \mathbf{b}_{k-1}. && \text{(i)} \end{aligned}$$

(a) Here we use the equality $\mathbf{b}_{k,k} = x_{k-1}x_k$.

(b) Here we modify the subword $y_k x_k y_{k+1} x_{k-1} x_k y_k$ by performing the substitution $(x, t, y, z) \mapsto (y_k, 1, x_k, y_{k+1}x_{k-1})$ in σ_2.

(c) Here we perform the substitution $(x, x_k) \mapsto (y_{k+1}, y_k x_k)$ in (6.4) and use the equality $\mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}$.

(d) Here we modify the subword $y_k x_k x_{k-1} y_k x_k$ by performing the substitution

$$(x, t, y, z) \mapsto (y_k, 1, x_k, x_{k-1})$$

in σ_2.

(e) Here we use the equality $\mathbf{b}_{k-1} = x_{k-2}x_{k-1}\mathbf{b}_{k-2}$.

(f) Here (6.3) allows us to add two new occurrences of the letter x_k after its second occurrence in the word $y_{k+1}^2 y_k \overset{(1)}{x_k} x_{k-1} \overset{(2)}{x_k} y_k x_{k-2} x_{k-1} \mathbf{b}_{k-2}$.

(g) Here we perform the substitution $(x, x_k, x_{k-1}) \mapsto (y_{k+1}, y_k, x_k x_{k-1} x_k)$ in (6.4) and use the equality $\mathbf{b}_k = x_{k-1}x_k x_{k-2} x_{k-1}\mathbf{b}_{k-2}$.

(h) Here (6.3) allows us to delete the third and fourth occurrences of the letter x_k in the word $y_{k+1} y_k y_{k+1} \overset{(1)}{x_k} x_{k-1} \overset{(2)}{x_k} y_k x_{k-2} \overset{(3)}{x_k} x_{k-1} \overset{(4)}{x_k} \mathbf{b}_{k-2}$.

(i) Here we use the equalities $\mathbf{b}_{k,k} = x_{k-1}x_k$ and $\mathbf{b}_{k-1} = x_{k-2}x_{k-1}\mathbf{b}_{k-2}$. ∎

Lemma 6.4. *We have*

$$\mathbf{F}_k \subseteq \mathbf{H}_k \subseteq \mathbf{I}_k \subseteq \mathbf{J}_k^1 \subseteq \mathbf{J}_k^2 \subseteq \cdots \subseteq \mathbf{J}_k^k \subseteq \mathbf{F}_{k+1}. \tag{6.5}$$

Proof. Since all varieties that appear in (6.5) are contained in $\mathbf{K}$, we can apply Lemma 6.2. In particular, this allows us to use the identities σ_2 and (6.3).

1°. *The inclusion* $\mathbf{F}_k \subseteq \mathbf{H}_k$. We need to verify that β_k follows from Φ and α_k. Here is the required deduction:

$$\begin{aligned}
x x_k x \mathbf{b}_k &= x x_k x x_{k-1} x_k \mathbf{b}_{k-1} && \text{because } \mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}\\
&\approx x x_k x x_{k-1} x_k x^2 \mathbf{b}_{k-1} && \text{by (6.3)}\\
&\approx x_k x^2 x_{k-1} x_k x^2 \mathbf{b}_{k-1} && \text{we perform the substitution}\\
& && (x_k, y_k) \mapsto (x_k x, x) \text{ in } \alpha_k\\
&\approx x_k x^2 x_{k-1} x_k \mathbf{b}_{k-1} && \text{by (6.3)}\\
&= x_k x^2 \mathbf{b}_k && \text{because } \mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}.
\end{aligned}$$

2°. *The inclusion* $\mathbf{H}_k \subseteq \mathbf{I}_k$. Here we need to verify that γ_k follows from Φ and β_k. Indeed,

$$\begin{aligned}
y_1 y_0 x_k y_1 \mathbf{b}_k &\approx y_1 y_0 x_k y_1^2 \mathbf{b}_k && \text{by (6.3)}\\
&\approx y_1 y_0 y_1 x_k y_1 \mathbf{b}_k && \text{we modify the subword } x_k y_1^2 \mathbf{b}_k\\
& && \text{by substituting } y_1 \text{ for } x \text{ in } \beta_k\\
&\approx y_1 y_0 y_1 x_k \mathbf{b}_k && \text{by (6.3).}
\end{aligned}$$

3°. *The inclusion* $\mathbf{I}_k \subseteq \mathbf{J}_k^1$. It suffices to verify that δ_k^1 follows from γ_k. Since $\mathbf{b}_{k,1} = \mathbf{b}_k$ and $\mathbf{b}_0 = \lambda$, the identity δ_k^1 has the form

$$y_2 y_1 x_k y_2 \mathbf{b}_k y_1 \approx y_2 y_1 y_2 x_k \mathbf{b}_k y_1.$$

To deduce this identity from γ_k, it suffices to modify the subword $y_2 y_1 x_k y_2 \mathbf{b}_k$ by performing the substitution $(y_0, y_1) \mapsto (y_1, y_2)$ in γ_k.

4°. *The inclusion* $\mathbf{J}_k^m \subseteq \mathbf{J}_k^{m+1}$ *where* $1 \le m < k$. It suffices to verify that δ_k^{m+1} follows from δ_k^m. Indeed, we get δ_k^{m+1} if we multiply δ_k^m by $x_{-1}x_0$ on the left and then increase by 1 the index of each letter in the resulting identity.

5°. *The inclusion* $\mathbf{J}_k^k \subseteq \mathbf{F}_{k+1}$. In view of Lemma 6.3, it suffices to verify that α_{k+1} follows from Φ and (6.4). We have

$$\begin{aligned} x_{k+1}y_{k+1}x_kx_{k+1}y_{k+1}\mathbf{b}_k &\approx (x_{k+1}y_{k+1})^2x_k\mathbf{b}_k && \text{(a)}\\ &\approx (y_{k+1}x_{k+1})^2x_k\mathbf{b}_k && \text{(b)}\\ &\approx y_{k+1}x_{k+1}x_ky_{k+1}x_{k+1}\mathbf{b}_k && \text{(c)}\\ &\approx y_{k+1}x_{k+1}x_kx_{k+1}y_{k+1}\mathbf{b}_k. && \text{(d)} \end{aligned}$$

(a) Here we substitute x_ky_k for x in (6.4).
(b) Here we apply the identity $(xy)^2 \approx (yx)^2$ that holds in $\mathbf{K}$ by Lemma 6.2(iii).
(c) Here we substitute y_kx_k for x in (6.4).
(d) Here we perform the substitution $(x,t,y,z) \mapsto (y_{k+1}, 1, x_{k+1}, x_k)$ in σ_2. ■

Below we often use the inclusions (6.5) without reference to Lemma 6.4. Note that in fact strict inclusions (6.1) are valid. We will prove these inclusions in Subsection 6.4.6.

6.2.2. k-decompositions of sides of the identities α_k, β_k, γ_k and δ_k^m

LEMMA 6.5. *Let* $\mathbf{u}$ *be the left-hand or the right-hand side of one of the identities* α_k, β_k, γ_k *or* δ_k^m. *Then:*

(1) *If* $x_i, y_j \in \operatorname{con}(\mathbf{u})$ *then* $D(\mathbf{u}, x_i) = i$ *and* $D(\mathbf{u}, y_j) = j$. *The depth of the letter* x *in the left-hand [right-hand] side of the identity* β_k *equals* $k+1$ *[respectively* ∞*].*
(2) *The* k*-decomposition of the word* $\mathbf{u}$ *has the form indicated in Table* 6.1.

As in Example 3.3, we underline k-blocks of words in Table 6.1 to distinguish them from k-dividers.

Table 6.1. k-decompositions of some words

The identity	The k-decomposition of the	
	left-hand side	right-hand side
α_k	$\lambda \cdot \underline{\lambda} \cdot x_k \cdot \underline{\lambda} \cdot y_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_k \cdot \underline{\lambda} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
β_k	$\lambda \cdot \underline{x} \cdot x_k \cdot \underline{x} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot x_k \cdot \underline{x^2} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
γ_k	$\lambda \cdot \underline{\lambda} \cdot y_1 \cdot \underline{\lambda} \cdot y_0 \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_1} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_1 \cdot \underline{\lambda} \cdot y_0 \cdot \underline{y_1} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
δ_k^m with $m < k$	$\lambda \cdot \underline{\lambda} \cdot y_{m+1} \cdot \underline{\lambda} \cdot y_m \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_{m+1}} \cdot x_{k-1} \cdot \underline{x_k} \cdots x_{m-1} \cdot \underline{x_m y_m} \cdot x_{m-2} \cdot \underline{x_{m-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_{m+1} \cdot \underline{\lambda} \cdot y_m \cdot \underline{y_{m+1}} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k} \cdots x_{m-1} \cdot \underline{x_m y_m} \cdot x_{m-2} \cdot \underline{x_{m-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
δ_k^k	$\lambda \cdot \underline{y_{k+1}} \cdot y_k \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_{k+1}} \cdot x_{k-1} \cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{y_{k+1}} \cdot y_k \cdot \underline{y_{k+1}} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$

Proof of Lemma 6.5. We restrict ourselves to verifying both the claims for the left-hand side of α_k only. In all other cases the proof is similar. We denote the left-hand side of α_k by $\mathbf{u}_k$. So,

$$\mathbf{u}_k = x_k y_k x_{k-1} x_k y_k x_{k-2} x_{k-1} x_{k-3} x_{k-2} \cdots x_1 x_2 x_0 x_1.$$

(1) The letter x_0 is simple in $\mathbf{u}_k$, whence $D(\mathbf{u}_k, x_0) = 0$. All other letters from $\mathrm{con}(\mathbf{u}_k)$ occur in $\mathbf{u}_k$ exactly twice. In particular, they are multiple in $\mathbf{u}_k$, and therefore their depth in $\mathbf{u}_k$ is greater than 0. The first occurrence of x_1 in $\mathbf{u}_k$ is not preceded by any simple letter. Therefore, $h_1^0(\mathbf{u}_k, x_1) = \lambda$. Further, only x_0 is simple in $\mathbf{u}_k$ and precedes the second occurrence of x_1 in $\mathbf{u}_k$. Hence $h_2^0(\mathbf{u}_k, x_1) = x_0$. We see that $h_1^0(\mathbf{u}_k, x_1) \neq h_2^0(\mathbf{u}_k, x_1)$, whence $D(\mathbf{u}_k, x_1) = 1$.

Neither the first nor the second occurrence of x_2 in $\mathbf{u}_k$ is preceded by any letter simple in $\mathbf{u}_k$. This means that $h_1^0(\mathbf{u}_k, x_2) = h_2^0(\mathbf{u}_k, x_2) = \lambda$, whence $D(\mathbf{u}_k, x_2) > 1$. The second occurrence of x_2 in $\mathbf{u}_k$ is preceded by exactly one occurrence of x_1 and there are no letters between these occurrences of x_1 and x_2. Moreover, $h_1^0(\mathbf{u}_k, x_1) \neq h_2^0(\mathbf{u}_k, x_1)$. Therefore, $h_2^1(\mathbf{u}_k, x_2) = x_1$. On the other hand, $h_1^1(\mathbf{u}_k, x_2) \neq x_1$ because x_1 does not occur before the first occurrence of x_2 in $\mathbf{u}_k$. Thus, $h_1^1(\mathbf{u}_k, x_2) \neq h_2^1(\mathbf{u}_k, x_2)$, whence $D(\mathbf{u}_k, x_2) = 2$.

We introduce some new notation to facilitate further considerations. For $a \in \mathrm{mul}(\mathbf{u}_k)$, we denote by $\mathbf{u}_k[a; 1, 2]$ the subword of $\mathbf{u}_k$ between the first and the second occurrences of a in $\mathbf{u}_k$. For instance, $\mathbf{u}_k[x_k; 1, 2] = y_k x_{k-1}$, $\mathbf{u}_k[y_k; 1, 2] = x_{k-1} x_k$, while $\mathbf{u}_k[x_1; 1, 2] = x_2 x_0$. Let now $2 < r < k$. Suppose that we have proved $D(\mathbf{u}_k, x_i) = i$ for all $i = 0, 1, \ldots, r-1$. We are going to check that $D(\mathbf{u}_k, x_r) = r$. Suppose that $D(\mathbf{u}_k, x_r) = s < r$. This means that $h_1^{s-1}(\mathbf{u}_k, x_r) \neq h_2^{s-1}(\mathbf{u}_k, x_r)$. Therefore, there is a letter z such that its first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r; 1, 2]$ and $h_1^{s-2}(\mathbf{u}_k, z) \neq h_2^{s-2}(\mathbf{u}_k, z)$. But $\mathbf{u}_k[x_r; 1, 2] = x_{r+1} x_{r-1}$ whenever $r < k-1$ and $\mathbf{u}_k[x_{k-1}; 1, 2] = x_k y_k x_{k-2}$. In any case, the unique letter whose first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r; 1, 2]$ is x_{r-1}. In view of our assumption, $D(\mathbf{u}_k, x_{r-1}) = r-1$. Since $s-2 < r-2$, the last equality implies that $h_1^{s-2}(\mathbf{u}_k, x_{r-1}) = h_2^{s-2}(\mathbf{u}_k, x_{r-1})$. Thus, there are no letters z with the above-mentioned properties. Therefore, $D(\mathbf{u}_k, x_r) \geq r$. Suppose now that $D(\mathbf{u}_k, x_r) = t > r$. Then $h_1^{r-1}(\mathbf{u}_k, x_r) = h_2^{r-1}(\mathbf{u}_k, x_r)$. Therefore, there are no letters z whose first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r; 1, 2]$ and $D(\mathbf{u}_k, z) = r-1$. But our assumption implies that x_{r-1} has these properties. Thus, $D(\mathbf{u}_k, x_r) = r$.

Quite analogous arguments establish that $D(\mathbf{u}_k, y_k) = k$. One has to take into account the equality $D(\mathbf{u}_k, x_{k-1}) = k-1$ proved above and the fact that the unique letter whose first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}[y_k; 1, 2]$ is x_{k-1}.

It remains to verify that $D(\mathbf{u}_k, x_k) = k$. We note that neither the first nor the second occurrence of x_k in $\mathbf{u}_k$ is preceded by any simple letter, whence $h_1^0(\mathbf{u}_k, x_k) = h_2^0(\mathbf{u}_k, x_k) = \lambda$. Suppose that $h_1^i(\mathbf{u}_k, x_k) \neq h_2^i(\mathbf{u}_k, x_k)$ for some $0 < i < k-1$. Then there is a letter z such that its first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}[x_k; 1, 2]$ and $h_1^{i-1}(\mathbf{u}_k, z) = h_2^{i-1}(\mathbf{u}_k, z)$. This means that $D(\mathbf{u}_k, z) \leq i < k-1$. Further, $\mathbf{u}[x_k; 1, 2] = y_k x_{k-1}$ and occurrences of both y_k and x_{k-1} are the first occurrences of these letters in $\mathbf{u}_k$. As we have seen above, $D(\mathbf{u}_k, y_k), D(\mathbf{u}_k, x_{k-1}) \geq k-1$. Thus, $h_1^i(\mathbf{u}_k, x_k) = h_2^i(\mathbf{u}_k, x_k)$ for all $0 \leq i < k-1$. Now we check that $h_1^k(\mathbf{u}_k, x_k) \neq h_2^k(\mathbf{u}_k, x_k)$. Indeed, we have seen above that $D(\mathbf{u}_k, x_{k-1}) = k-1$ and $D(\mathbf{u}_k, y_k) = k$. Therefore, $h_1^{k-2}(\mathbf{u}_k, x_{k-1}) \neq h_2^{k-2}(\mathbf{u}_k, x_{k-1})$ and

$h_1^{k-2}(\mathbf{u}_k, y_k) = h_2^{k-2}(\mathbf{u}_k, y_k)$. This implies that $h_2^{k-1}(\mathbf{u}_k, x_k) = x_{k-1}$. On the other hand, the first occurrence of x_k in $\mathbf{u}_k$ is not preceded by any letter, whence $h_1^{k-1}(\mathbf{u}_k, x_k) = \lambda$. We see that $h_1^k(\mathbf{u}_k, x_k) \ne h_2^k(\mathbf{u}_k, x_k)$. In view of the above, this means that $D(\mathbf{u}_k, x_k) = k$.

(2) By Lemma 3.7, the k-dividers of a word $\mathbf{w}$ are exactly the first occurrences of letters $x \in \mathrm{con}(\mathbf{w})$ with $D(\mathbf{w}, x) \le k$ and the empty word at the beginning of the word $\mathbf{w}$. As proved above, $D(\mathbf{u}_k, x) \le k$ for any $x \in \mathrm{con}(\mathbf{u}_k)$. Thus, the k-dividers of $\mathbf{u}_k$ are just the first occurrences of all letters from $\mathrm{con}(\mathbf{u}_k)$ and the empty word at the beginning of $\mathbf{u}_k$. All subwords of $\mathbf{u}_k$ between these k-dividers and only they are k-blocks of $\mathbf{u}_k$. Thus, the k-decomposition of the word $\mathbf{u}_k$ has the form indicated in Table 6.1. ∎

Note that claim (1) of Lemma 6.5 explains the choice of indices of letters in the identities α_k, β_k, γ_k and δ_k^m.

6.2.3. Swapping letters within k-blocks. In this subsection we verify only one statement. It is the core of the whole proof of Theorem 1.1. Its proof is very long and technical. At the same time, it is the basis for the rest of the proof of Theorem 1.1 and plays a key role there.

LEMMA 6.6. *Let $\mathbf{V}$ be a monoid variety such that $\mathbf{V} \subseteq \mathbf{K}$, $\mathbf{u}$ be a word and k be a natural number. Further, let $\mathbf{u} = \mathbf{u}'ab\mathbf{u}''$ where $\mathbf{u}'$ and $\mathbf{u}''$ are possibly empty words, while ab is a subword of some $(k-1)$-block of $\mathbf{u}$. Suppose that one of the following holds:*

(i) *$\mathbf{V}$ satisfies δ_k^m, $a \in \mathrm{con}(\mathbf{u}')$ and $D(\mathbf{u}, a) > m$;*
(ii) *$\mathbf{V}$ satisfies γ_k and $a \in \mathrm{con}(\mathbf{u}')$;*
(iii) *$\mathbf{V}$ satisfies β_k and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$;*
(iv) *$\mathbf{V}$ satisfies α_k.*

Then $\mathbf{V}$ satisfies the identity $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$.

Proof. We will prove assertions (i)–(iv) simultaneously. Suppose that $\mathbf{V}$ satisfies the hypothesis of one of these four claims. In particular, $\mathbf{V}$ satisfies δ_k^k in any case. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$ and ab is a subword of $\mathbf{u}_i$ for some $0 \le i \le m$. Then $\mathbf{u}_i = \mathbf{u}_i'ab\mathbf{u}_i''$ for some possibly empty words $\mathbf{u}_i'$ and $\mathbf{u}_i''$. Clearly, $\mathbf{u}' = t_0\mathbf{u}_0t_1\mathbf{u}_1 \cdots t_i\mathbf{u}_i'$ and $\mathbf{u}'' = \mathbf{u}_i''t_{i+1}\mathbf{u}_{i+1} \cdots t_m\mathbf{u}_m$.

If $a, b \in \mathrm{con}(\mathbf{u}')$ then

$$\mathbf{u} = \mathbf{u}'ab\mathbf{u}'' \stackrel{(4.9)}{\approx} \mathbf{u}'a^2b^2\mathbf{u}'' \stackrel{(4.4)}{\approx} \mathbf{u}'b^2a^2\mathbf{u}'' \stackrel{(4.9)}{\approx} \mathbf{u}'ba\mathbf{u}'',$$

and we are done. Thus, we can assume without loss of generality that

$$b \notin \mathrm{con}(\mathbf{u}'). \tag{6.6}$$

If $D(\mathbf{u}, b) \le k-1$ then b is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case because the first occurrence of b in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$. Therefore, $D(\mathbf{u}, b) \ge k$. Further, if $a \in \mathrm{mul}(\mathbf{u}')$ then Lemma 6.2(ii) implies that the identities $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'b\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ hold in $\mathbf{V}$. Thus, we can assume that

$$\text{if } a \in \mathrm{con}(\mathbf{u}') \text{ then } a \in \mathrm{sim}(\mathbf{u}'). \tag{6.7}$$

Further considerations are divided into three cases depending on the depth of b in $\mathbf{u}$: $D(\mathbf{u}, b) = k$, $k < D(\mathbf{u}, b) < \infty$ and $D(\mathbf{u}, b) = \infty$. Each of these cases is divided into subcases corresponding to claims (i)–(iv). Thus, the proof of each of assertions (i)–(iv) will be completed after considering the corresponding subcase of Case 3.

Case 1: $D(\mathbf{u}, b) = k$. This case is the most difficult from the technical point of view and the longest. By examining two other cases, we will repeatedly refer to properties that will be verified here. Let $\mathbf{p} \approx \mathbf{q}$ be one of the identities α_k, β_k, γ_k or δ_k^m. In a sense, the identity $\mathbf{p} \approx \mathbf{q}$ "looks like" $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$. We have in mind that the words $\mathbf{p}$ and $\mathbf{q}$ start with the same prefix (which is empty for α_k and β_k) and end with the same suffix, and the subword between the prefix and the suffix is the product of two letters in $\mathbf{p}$ and the product of the same two letters in the reverse order in $\mathbf{q}$. This makes it possible in principle to apply the identity $\mathbf{p} \approx \mathbf{q}$ to one of the sides of the identity $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ in order to obtain the other side of it. To realize this possibility, we need, with the use of the identities that hold in $\mathbf{K}$, to reduce, say, the right-hand side of the identity $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ to a form to which the identity $\mathbf{p} \approx \mathbf{q}$ can be applied. To do this, we first need to find "inside" $\mathbf{u}$ the letters $x_0, x_1, \dots, x_k$ which would appear in the same order as the letters with the same names in one of the sides of the identity $\mathbf{p} \approx \mathbf{q}$.

Put $x_k = b$. Let X_{k-1} be the set of $(k-1)$-dividers z of $\mathbf{u}$ such that

$$\ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_k).$$

The fact that $D(\mathbf{u}, x_k) = k$ implies $h_1^{k-1}(\mathbf{u}, x_k) \ne h_2^{k-1}(\mathbf{u}, x_k)$, whence $h_2^{k-1}(\mathbf{u}, x_k) \in X_{k-1}$. Therefore, X_{k-1} is non-empty. Further, Lemma 3.9(ii) implies that $D(\mathbf{u}, z) = k-1$ and $\ell_2(\mathbf{u}, x_k) < \ell_2(\mathbf{u}, z)$ for any $z \in X_{k-1}$. Now we consider the letter $x_{k-1} \in X_{k-1}$ such that $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, x_{k-1})$ for any $z \in X_{k-1}$.

Let X_{k-2} be the set of $(k-2)$-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u}, x_{k-1}) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{k-1})$. Then $D(\mathbf{u}, x_{k-1}) = k-1$ implies that $h_1^{k-2}(\mathbf{u}, x_{k-1}) \ne h_2^{k-2}(\mathbf{u}, x_{k-1})$, whence $h_2^{k-2}(\mathbf{u}, x_{k-1}) \in X_{k-2}$. Therefore, X_{k-2} is non-empty. Further, Lemma 3.9(ii) implies that $D(\mathbf{u}, z) = k-2$ and $\ell_2(\mathbf{u}, x_{k-1}) < \ell_2(\mathbf{u}, z)$ for any $z \in X_{k-2}$. Now we consider the letter $x_{k-2} \in X_{k-2}$ such that $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, x_{k-2})$ for any $z \in X_{k-2}$. Since $\ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, x_{k-1}) < \ell_1(\mathbf{u}, x_{k-2})$, Lemma 3.13 implies that $\ell_2(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, x_{k-2})$.

Further, for $s = k-3, k-4, \dots, 1$ we define one by one the set X_s and the letter x_s in the following way: X_s is the set of all s-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u}, x_{s+1}) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s+1})$, and x_s is a letter such that $x_s \in X_s$ and $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, x_s)$ for any $z \in X_s$. Arguments similar to those from the previous two paragraphs show that X_s is non-empty, $D(\mathbf{u}, x_s) = s$, $\ell_j(\mathbf{u}, x_{s+1}) < \ell_j(\mathbf{u}, x_s)$ for any $j = 1, 2$ and $\ell_2(\mathbf{u}, x_{s+2}) < \ell_1(\mathbf{u}, x_s)$.

Finally, put $x_0 = h_2^0(\mathbf{u}, x_1)$. In view of Lemma 3.9, we have $D(\mathbf{u}, x_0) = 0$ and $\ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$. Since $\ell_1(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_1)$, Lemma 3.13 implies $\ell_2(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_0)$. Then

$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 \tag{6.8}$$

for some possibly empty words $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2k}$. One can verify that if $2 \le s \le k$ then

$$\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1}) \quad \text{for any } z \in \mathrm{con}(\mathbf{v}_{2s}\mathbf{v}_{2s-1}). \tag{6.9}$$

Put

$$\mathbf{w}_s = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2s+2}x_s\mathbf{v}_{2s+1}x_{s+1}.$$

The word $\mathbf{w}_s$ is the prefix of $\mathbf{u}$ that immediately precedes $\mathbf{v}_{2s}$, while $\mathbf{v}_{2s-1}$ precedes the second occurrence of x_{s-1} in $\mathbf{u}$. This implies the required conclusion whenever $z \in \mathrm{con}(\mathbf{w}_s)$. Suppose now that $z \notin \mathrm{con}(\mathbf{w}_s)$. Then $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_s)$. If z is an $(s-1)$-divider of $\mathbf{u}$ then $z \in X_{s-1}$, whence $\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1})$ by the choice of x_{s-1}. Otherwise $D(\mathbf{u}, z) > s-1$ by Lemma 3.7. Then since $\ell_1(\mathbf{u}, z) < \ell_1(\mathbf{u}, x_{s-2})$, Lemma 3.13 implies that $\ell_2(\mathbf{u}, z) < \ell_1(\mathbf{u}, x_{s-2})$, whence $\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1})$.

The further realization of the plan outlined at the beginning of Case 1 depends on the identity that plays the role of $\mathbf{p} \approx \mathbf{q}$. Therefore, further considerations are divided into four subcases.

Subcase 1.1: $\mathbf{V}$ satisfies the hypothesis of claim (i), i.e., δ_k^m holds in $\mathbf{V}$, $a \in \mathrm{con}(\mathbf{u}')$ and $D(\mathbf{u}, a) > m$. Claim (6.7) allows us to assume that $a \in \mathrm{sim}(\mathbf{u}')$. Then $\mathbf{u}' = \mathbf{w}a\mathbf{v}$ for some possibly empty words $\mathbf{v}$ and $\mathbf{w}$. This implies that

$$\mathbf{u} = \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0. \tag{6.10}$$

Put $D(\mathbf{u}, a) = r$. Further considerations are divided into two parts corresponding to whether $r \le k+1$ or $r > k+1$.

(A) $r \le k+1$. Here we need to define two more letters, namely y_{r-1} and y_{r-2}, and clarify the location of these letters within $\mathbf{u}$. Let Y_{r-1} be the set of $(r-1)$-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, a)$. The fact that $D(\mathbf{u}, a) = r$ implies that $h_1^{r-1}(\mathbf{u}, a) \ne h_2^{r-1}(\mathbf{u}, a)$, whence $h_2^{r-1}(\mathbf{u}, a) \in Y_{r-1}$. Therefore, the set Y_{r-1} is non-empty. Lemma 3.9(ii) implies that $D(\mathbf{u}, z) = r-1$ and $\ell_2(\mathbf{u}, a) < \ell_2(\mathbf{u}, z)$ for any $z \in Y_{r-1}$. Then $\ell_1(\mathbf{u}, b) < \ell_2(\mathbf{u}, z)$ for any $z \in Y_{r-1}$. Now we consider the letter $y_{r-1} \in Y_{r-1}$ such that $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, y_{r-1})$ for any $z \in Y_{r-1}$.

Now we check some additional properties of the letter x_r, which are fulfilled under certain restrictions to r. Suppose that $r < k+1$. Then x_r is defined. Our aim is to prove that

$$\ell_2(\mathbf{u}, x_r) < \ell_2(\mathbf{u}, y_{r-1}). \tag{6.11}$$

Put $y_{r-2} = h_2^{r-2}(\mathbf{u}, y_{r-1})$. Since $D(\mathbf{u}, y_{r-1}) = r-1$, Lemma 3.9 implies that $D(\mathbf{u}, y_{r-2}) = r-2$ and $\ell_1(\mathbf{u}, y_{r-1}) < \ell_1(\mathbf{u}, y_{r-2})$. Recall that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, y_{r-1})$, which implies $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, y_{r-2})$. Since $D(\mathbf{u}, a) = r$, we can apply Lemma 3.13 to conclude that $\ell_2(\mathbf{u}, a) < \ell_1(\mathbf{u}, y_{r-2})$. The second occurrence of a in $\mathbf{u}$ immediately precedes the first occurrence of $b = x_k$, whence $\ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, y_{r-2})$. Then Lemma 3.13 implies that $\ell_2(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, y_{r-2})$. This yields $\ell_1(\mathbf{u}, x_{k-1}) < \ell_2(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, y_{r-2})$. If $k-1 \ge r$ then Lemma 3.13 shows that $\ell_2(\mathbf{u}, x_{k-1}) < \ell_1(\mathbf{u}, y_{r-2})$. Continuing, we eventually obtain $\ell_2(\mathbf{u}, x_r) < \ell_1(\mathbf{u}, y_{r-2})$. The choice of y_{r-2} implies that the first occurrence of y_{r-2} in $\mathbf{u}$ precedes the second occurrence of y_{r-1}. Therefore, $\ell_2(\mathbf{u}, x_r) < \ell_2(\mathbf{u}, y_{r-1})$. So, we have proved that if $r < k+1$ then (6.11) is true.

Let now $r > 2$. Note that

$$\ell_1(\mathbf{u}, y_{r-1}) < \ell_2(\mathbf{u}, a) < \ell_1(\mathbf{u}, b) = \ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, x_{k-1}) < \cdots < \ell_1(\mathbf{u}, x_{r-3}).$$

If $\ell_1(\mathbf{u}, x_{r-3}) < \ell_2(\mathbf{u}, y_{r-1})$ then x_{r-3} lies between the first and the second occurrences of y_{r-1} in $\mathbf{u}$. Since x_{r-3} is an $(r-3)$-divider of $\mathbf{u}$, we obtain a contradiction with the equality $D(\mathbf{u}, y_{r-1}) = r - 1$. Thus,

$$\ell_2(\mathbf{u}, y_{r-1}) < \ell_1(\mathbf{u}, x_{r-3}) \tag{6.12}$$

whenever $r > 2$.

One can return to arbitrary $r \le k+1$. This restriction guarantees that x_{r-2} and x_{r-1} are defined. There are three possibilities for the second occurrence of y_{r-1} in $\mathbf{u}$:

$$\ell_1(\mathbf{u}, x_{r-2}) < \ell_2(\mathbf{u}, y_{r-1}) < \ell_2(\mathbf{u}, x_{r-1}); \tag{6.13}$$

$$\ell_2(\mathbf{u}, y_{r-1}) < \ell_1(\mathbf{u}, x_{r-2}); \tag{6.14}$$

$$\ell_2(\mathbf{u}, x_{r-1}) < \ell_2(\mathbf{u}, y_{r-1}). \tag{6.15}$$

The equality (6.10) may be rewritten in the form

$$\begin{aligned}\mathbf{u} = \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}\,\overset{(1)}{x_{r-1}}\\ \cdot\,\mathbf{v}_{2r-1}\,\overset{(2)}{x_r}\,\mathbf{v}_{2r-2}\,\overset{(1)}{x_{r-2}}\,\mathbf{v}_{2r-3}\,\overset{(2)}{x_{r-1}}\,\mathbf{v}_{2r-4}\,\overset{(1)}{x_{r-3}}\,\mathbf{v}_{2r-5}\,\overset{(2)}{x_{r-2}}\cdots\\ \cdot\,\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned} \tag{6.16}$$

Suppose that (6.13) holds. Then the second occurrence of y_{r-1} in $\mathbf{u}$ belongs to $\mathbf{v}_{2r-3}$, whence $\mathbf{v}_{2r-3} = \mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}$ for possibly empty words $\mathbf{v}'_{2r-3}$ and $\mathbf{v}''_{2r-3}$. Further, since $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, y_{r-1}) < \ell_2(\mathbf{u}, a)$, the first occurrence of y_{r-1} belongs to $\mathbf{v}$. Therefore, $\mathbf{v} = \mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ for possibly empty words $\mathbf{v}_{2k+2}$ and $\mathbf{v}_{2k+1}$.

Combining all the above, we can clarify the presentation (6.10) of the word $\mathbf{u}$ and write this word in the form

$$\begin{aligned}\mathbf{u} = \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\\ \cdot\,\mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}x_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots\\ \cdot\,\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

Note that $\mathbf{u}' = \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ and

$$\begin{aligned}\mathbf{u}'' = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\\ \cdot\,\mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}x_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

Similarly to the proof of (6.9), we can verify that if $z \in \mathrm{con}(\mathbf{v}_{2k+2}\mathbf{v}_{2k+1})$ then $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, y_{r-1})$.

Now we are ready to begin the process of modifying $\mathbf{u}$ to get $\mathbf{u}'ba\mathbf{u}''$. But first, we will outline the general scheme of further considerations, since arguments of that type will be repeated many times below. We rely on the fact that (6.3) is satisfied by the variety $\mathbf{K}$. This allows us to add any letter that is multiple in a given word to any place after the second occurrence of this letter in the word. Using this, we will add different missing letters or even words in different places in $\mathbf{u}$ (or in a word which equals $\mathbf{u}$ in $\mathbf{V}$) in order to make it possible to apply to that word the identity that is fulfilled in $\mathbf{V}$ at the moment (now the identity is δ_k^m). Next, we will apply this identity, and then "reverse the process", i.e., making use of (6.3), remove unnecessary letters or even words from the resulting word to obtain $\mathbf{u}'ba\mathbf{u}''$.

Let us implement this plan. First, we apply (6.3) to $\mathbf{u}$ and insert y_{r-1} after the second occurrence of x_{r-1} in $\mathbf{u}$. We obtain

$$\begin{aligned}\mathbf{u} \approx \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots \\ \cdot \mathbf{v}_{2r}\overset{(1)}{x_{r-1}}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3}\overset{(2)}{x_{r-1}}y_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots \\ \cdot \mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned} \tag{6.17}$$

Further, we apply (6.3) sufficiently many times to the right-hand side of (6.17) and replace there the third occurrence of y_{r-1} with $\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ and the second occurrence of x_{s-1} with $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We find that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{p}\mathbf{v}_0 \tag{6.18}$$

where

$$\begin{aligned}\mathbf{p} = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\mathbf{v}_{2k-4}\cdots\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3} \\ \cdot \mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3}\mathbf{v}_{2r-6}\cdots \\ \cdot \mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3.\end{aligned}$$

By the hypothesis, $r = D(\mathbf{u}, a) > m$. Then by Lemma 6.4, δ_k^{r-1} holds in $\mathbf{V}$. Now we perform the substitution

$$(x_0, \dots, x_{k-1}, x_k, y_{r-1}, y_r) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, b, \mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}, a)$$

in δ_k^{r-1} to obtain the identity

$$a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{p} \approx a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ba\mathbf{p}.$$

This identity together with (6.18) implies that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ba\mathbf{p}\mathbf{v}_0.$$

Now we apply (6.3) to the right-hand side of the last identity "in the opposite direction" and replace $\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ with y_{r-1} and $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\begin{aligned}\mathbf{u} \approx \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r-2}x_{r-2} \\ \cdot \mathbf{v}_{2r-3}x_{r-1}y_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\mathbf{v}_{2r-6}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

Finally, we apply (6.3) to the right-hand side of the last identity and delete the third occurrence y_{r-1}. We obtain the identity

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r-2}x_{r-2} \\ &\quad \cdot \mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}x_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\mathbf{v}_{2r-6}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 \\ &= \mathbf{u}'ba\mathbf{u}''.\end{aligned}$$

It remains to consider the case when either (6.14) or (6.15) holds. We are going to verify that in both cases, (6.17) holds. This suffices because then we can complete our considerations as above. If (6.14) holds then (6.11) and (6.16) imply that

$$\begin{aligned}\mathbf{u} = \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}\overset{(1)}{x_{r-1}}\mathbf{v}_{2r-1}x_r\mathbf{v}'_{2r-2} \\ \cdot y_{r-1}\mathbf{v}''_{2r-2}x_{r-2}\mathbf{v}_{2r-3}\overset{(2)}{x_{r-1}}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{v}'_{2r-2}, \mathbf{v}''_{2r-2}$ such that $\mathbf{v}_{2r-2} = \mathbf{v}'_{2r-2} y_{r-1} \mathbf{v}''_{2r-2}$. Here we add one more occurrence of y_{r-1} immediately after the second occurrence of x_{r-1}. As a result, we obtain (6.17). Finally, if (6.15) is the case then we use (6.12). Then

$$\begin{aligned}\mathbf{u} = \mathbf{w} a \mathbf{v}_{2k+2} \overset{(1)}{y_{r-1}} \mathbf{v}_{2k+1} ab \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2r} x_{r-1} \mathbf{v}_{2r-1} x_r \\ \cdot \mathbf{v}_{2r-2} x_{r-2} \mathbf{v}_{2r-3} x_{r-1} \mathbf{v}'_{2r-4} \overset{(2)}{y_{r-1}} \mathbf{v}''_{2r-4} \overset{(1)}{x_{r-3}} \mathbf{v}_{2r-5} x_{r-2} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{v}'_{2r-4}, \mathbf{v}''_{2r-4}$ such that $\mathbf{v}_{2r-4} = \mathbf{v}'_{2r-4} y_{r-1} \mathbf{v}''_{2r-4}$. Then we can add a third occurrence of the letter x_{r-1} immediately before the second occurrence of y_{r-1} and obtain

$$\begin{aligned}\mathbf{u} \approx \mathbf{w} a \mathbf{v}_{2k+2} \overset{(1)}{y_{r-1}} \mathbf{v}_{2k+1} ab \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2r} x_{r-1} \mathbf{v}_{2r-1} x_r \mathbf{v}_{2r-2} \\ \cdot x_{r-2} \mathbf{v}_{2r-3} x_{r-1} \mathbf{v}'_{2r-4} x_{r-1} \overset{(2)}{y_{r-1}} \mathbf{v}''_{2r-4} \overset{(1)}{x_{r-3}} \mathbf{v}_{2r-5} x_{r-2} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.\end{aligned}$$

This is nothing but (6.17) (up to renaming of words).

(B) $r > k+1$. Recall that equality (6.10) is true. Suppose that $\mathbf{v}$ is non-empty. Let $y \in \mathrm{con}(\mathbf{v})$. Suppose that $\ell_1(\mathbf{u}, x_{k-1}) < \ell_2(\mathbf{u}, y)$. This implies that $h_1^{k-1}(\mathbf{u}, y) \neq h_2^{k-1}(\mathbf{u}, y)$ because x_{k-1} is a $(k-1)$-divider of $\mathbf{u}$. Then y is a k-divider of $\mathbf{u}$. Since $\mathbf{v}$ (and in particular y) is located between the first and the second occurrences of a in $\mathbf{u}$, this contradicts $D(\mathbf{u}, a) = r > k+1$. So, $\ell_2(\mathbf{u}, y) \le \ell_1(\mathbf{u}, x_{k-1})$ for any $y \in \mathrm{con}(\mathbf{v})$. Then we apply (6.3) sufficiently many times to the right-hand side of (6.10), namely, we insert $\mathbf{v}$ after the second occurrence of b there. Clearly, we can formally insert $\mathbf{v}$ after the second occurrence of b whenever $\mathbf{v} = \lambda$ too. Further, in view of (6.9), we can replace the second occurrence of x_{s-1} in the right-hand side of (6.10) with $\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We deduce that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w} a \mathbf{v} ab \mathbf{p} \mathbf{v}_0 \tag{6.19}$$

where

$$\mathbf{p} = \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v} \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 \mathbf{v}_6 x_2 \mathbf{v}_5 \mathbf{v}_2 x_0 \mathbf{v}_1 \mathbf{v}_4 x_1 \mathbf{v}_3.$$

In view of Lemma 6.4, $\mathbf{V}$ satisfies δ_k^k. Now we perform the substitution

$$(x_0, \dots, x_{k-1}, x_k, y_k, y_{k+1}) \mapsto (\mathbf{v}_2 x_0 \mathbf{v}_1, \dots, \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1}, b, \mathbf{v}, a)$$

in δ_k^k to obtain the identity

$$a \mathbf{v} ab \mathbf{p} \approx a \mathbf{v} ba \mathbf{p}.$$

This identity together with (6.19) implies that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w} a \mathbf{v} ba \mathbf{p} \mathbf{v}_0.$$

Now we apply (6.3) to the right-hand side of the last identity "in the opposite direction", namely we delete $\mathbf{v}$ after the second occurrence of b and replace $\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\mathbf{u} \approx \mathbf{w} a \mathbf{v} ba \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0 = \mathbf{u}' ba \mathbf{u}''.$$

Subcase 1.2: $\mathbf{V}$ satisfies the hypothesis of claim (ii), i.e., γ_k holds in $\mathbf{V}$ and $a \in \mathrm{con}(\mathbf{u}')$. Recall that equality (6.8) is true. Claim (6.7) allows us to assume that $a \in \mathrm{sim}(\mathbf{u}')$. Then, as in Subcase 1.1, $\mathbf{u}$ has the form (6.10). Note that $\mathbf{u}' = \mathbf{w}a\mathbf{v}$ and

$$\mathbf{u}'' = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Recall that (6.9) is true for any $2 \le s \le k$. Now we can apply (6.3) sufficiently many times to the right-hand side of (6.10) and replace the second occurrence of x_{s-1} with $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We infer that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots$$
$$\cdot\ \mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_0.$$

Put $\mathbf{p}_1 = a\mathbf{v}$ and

$$\mathbf{p}_2 = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3.$$

Then the last identity has the form

$$\mathbf{u} \approx \mathbf{w}\mathbf{p}_1ab\mathbf{p}_2\mathbf{v}_0. \tag{6.20}$$

By the hypothesis, $\mathbf{V}$ satisfies the identity γ_k. Now we perform the substitution

$$(x_0, x_1, \dots, x_{k-1}, x_k, y_0, y_1) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \mathbf{v}_4x_1\mathbf{v}_3, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, b, \mathbf{v}, a)$$

in γ_k to obtain the identity $\mathbf{p}_1ba\mathbf{p}_2 \approx \mathbf{p}_1ab\mathbf{p}_2$. This identity together with (6.20) implies that $\mathbf{V}$ satisfies $\mathbf{u} \approx \mathbf{w}\mathbf{p}_1ba\mathbf{p}_2\mathbf{v}_0$, i.e.,

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots$$
$$\cdot\ \mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_0.$$

Now we apply (6.3) to the right-hand side of the last identity "in the opposite direction" and replace $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0,$$

i.e., $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$.

Subcase 1.3: $\mathbf{V}$ satisfies the hypothesis of claim (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Subcase 1.2 allows us to assume that $a \notin \mathrm{con}(\mathbf{u}')$. This fact and (6.6) immediately imply that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$. If $D(\mathbf{u}, a) \le k - 1$ then a is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case. Therefore, $D(\mathbf{u}, a) \ge k$. Since $D(\mathbf{u}, b) \ne D(\mathbf{u}, a)$ and $D(\mathbf{u}, b) = k$, we obtain $D(\mathbf{u}, a) > k$.

Note that $\ell_2(\mathbf{u}, a) < \ell_1(\mathbf{u}, x_{k-1})$ because $h_1^{k-1}(\mathbf{u}, a) = h_2^{k-1}(\mathbf{u}, a)$ and x_{k-1} is a $(k-1)$-divider. Recall that (6.8) is true. Then $\mathbf{v}_{2k} = \mathbf{v}'_{2k}a\mathbf{v}''_{2k}$ for some possibly empty words $\mathbf{v}'_{2k}, \mathbf{v}''_{2k}$. Thus,

$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}'_{2k}a\mathbf{v}''_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Now we are going to verify that the identity

$$\mathbf{u} \approx \mathbf{u}'aba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 \tag{6.21}$$

holds in $\mathbf{V}$. This is evident whenever $\mathbf{v}'_{2k} = \lambda$. Suppose now that $\mathbf{v}'_{2k} = \mathbf{v}^*d$ for some possibly empty word $\mathbf{v}^*$ and some letter d. Then $\mathbf{u}$ may be rewritten in the form

$$\mathbf{u} = \mathbf{u}'\ \overset{(1)}{a}\ b\mathbf{v}^*d\ \overset{(2)}{a}\ \mathbf{v}''_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Note that the subword da located between $\mathbf{v}^*$ and $\mathbf{v}''_{2k}$ lies in some $(k-1)$-block of $\mathbf{u}$. Indeed, the occurrence of d in this subword is not a $(k-1)$-divider of $\mathbf{u}$ because otherwise the first and the second occurrences of a in $\mathbf{u}$ lie in different $(k-1)$-blocks, contradicting the inequality $D(\mathbf{u},a) > k$, while the occurrence of a in this subword is not a $(k-1)$-divider of $\mathbf{u}$ because this is not the first occurrence of a in $\mathbf{u}$.

According to Lemma 6.4, the variety $\mathbf{V}$ satisfies the identity γ_k. In view of the statement that was proved in Subcase 1.2, $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'ab\mathbf{v}^*ad\mathbf{v}''_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Acting in this way, we can successively swap the letter a with all letters of the word $\mathbf{v}'_{2k}$ to deduce that

$$\mathbf{u} \approx \mathbf{u}'aba\mathbf{v}'_{2k}\mathbf{v}''_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0$$

holds in $\mathbf{V}$. Now we apply (6.3) to the right-hand side of the last identity and insert the letter a after $\mathbf{v}'_{2k}$. We obtain (6.21).

Recall that (6.9) is true for any $2 \le s \le k$. Now we can apply (6.3) sufficiently many times to the right-hand side of (6.21) and replace the second occurrence of x_{s-1} with $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We conclude that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'aba\mathbf{p}\mathbf{v}_0 \tag{6.22}$$

where

$$\mathbf{p} = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3.$$

Now we perform the substitution

$$(x_0, x_1, \ldots, x_{k-1}, x_k, x) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \mathbf{v}_4x_1\mathbf{v}_3, \ldots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, b, a)$$

in β_k, which yields $aba\mathbf{p} \approx ba^2\mathbf{p}$. One can apply this identity to (6.22). We find that the identity

$$\mathbf{u} \approx \mathbf{u}'ba^2\mathbf{p}\mathbf{v}_0 = \mathbf{u}'ba^2\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots$$
$$\cdot\, \mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_0$$

holds in $\mathbf{V}$. Now we apply (6.3) to the right-hand side of the last identity “in the opposite direction” and replace $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\mathbf{u} \approx \mathbf{u}'ba^2\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Repeating the arguments used above in the deduction of (6.21), we find that $\mathbf{V}$ satisfies

$$\mathbf{u} \approx \mathbf{u}'ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 = \mathbf{u}'ba\mathbf{u}'',$$

i.e., $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$.

Subcase 1.4: $\mathbf{V}$ satisfies the hypothesis of claim (iv), i.e., α_k holds in $\mathbf{V}$. By Subcases 1.2 and 1.3, and by (6.6), we can assume that $a, b \notin \mathrm{con}(\mathbf{u}')$ and $D(\mathbf{u},b) = D(\mathbf{u},a)$. Recall that (6.8) is true.

Note that $\ell_2(\mathbf{u},a) < \ell_1(\mathbf{u},x_{k-2})$ because $h_1^{k-2}(\mathbf{u},a) = h_2^{k-2}(\mathbf{u},a)$ and x_{k-2} is a $(k-2)$-divider. Therefore, there are possibly empty words $\mathbf{v}'$ and $\mathbf{v}''$ such that one of the following equalities holds:

$$\mathbf{v}_{2k} = \mathbf{v}'a\mathbf{v}'', \quad \mathbf{v}_{2k-1} = \mathbf{v}'a\mathbf{v}'' \quad \text{or} \quad \mathbf{v}_{2k-2} = \mathbf{v}'a\mathbf{v}''.$$

Then one of the following equalities holds:

$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}'a\mathbf{v}''x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0,$$
$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}'a\mathbf{v}''b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0,$$
$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}'a\mathbf{v}''x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

We consider only the first case; the other two cases can be considered similarly. Since $\mathbf{V}$ satisfies (6.3), we see that the identity

$$\mathbf{u} \approx \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 \tag{6.23}$$

holds in this variety.

Recall that (6.9) is true for any $2 \le s \le k$. Now we can apply (6.3) sufficiently many times to the right-hand side of (6.23) and replace the second occurrence of x_{s-1} with $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We deduce that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'ab\mathbf{p}\mathbf{v}_0 \tag{6.24}$$

where

$$\mathbf{p} = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3.$$

Now we perform the substitution

$$(x_0, x_1, \dots, x_{k-1}, x_k, y_k) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \mathbf{v}_4x_1\mathbf{v}_3, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, a, b)$$

in α_k to obtain $ab\mathbf{p} \approx ba\mathbf{p}$. Applying this identity to (6.24), we get $\mathbf{u} \approx \mathbf{u}'ba\mathbf{p}\mathbf{v}_0$. Now we apply (6.3) to the right-hand side of the last identity "in the opposite direction" and replace $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain

$$\mathbf{u} \approx \mathbf{u}'ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Now we apply (6.3) again and delete the occurrence of a located between $\mathbf{v}_{2k-1}$ and the second occurrence of b in the right-hand side of the last identity. We see that $\mathbf{V}$ satisfies

$$\mathbf{u}\approx\mathbf{u}'ba\mathbf{v}'a\mathbf{v}''x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0=\mathbf{u}'ba\mathbf{u}'',$$

i.e., $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$.

Case 2: $k < D(\mathbf{u}, b) < \infty$. As we will see below, this case reduces to the previous one by relatively simple arguments. Put $D(\mathbf{u}, b) = r$. Further considerations are divided into three subcases.

Subcase 2.1: $\mathbf{V}$ satisfies the hypothesis of (i) or (ii). Here $a \in \operatorname{con}(\mathbf{u}')$. Hence the occurrence of a in the subword ab of $\mathbf{u}$ mentioned in the formulation of the lemma is not the first occurrence of a in $\mathbf{u}$. Therefore, this occurrence of a in $\mathbf{u}$ is not an $(r-1)$-divider of $\mathbf{u}$. Lemma 3.7 together with the fact that $D(\mathbf{u}, b) = r$ implies that the occurrence of b in the subword ab of $\mathbf{u}$ is not an $(r-1)$-divider of $\mathbf{u}$ either. Therefore, ab lies in some $(r-1)$-block of $\mathbf{u}$.

Let

$$s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_n\mathbf{w}_n \tag{6.25}$$

be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists $0\le j\le n$ such that $\mathbf{w}_j=\mathbf{w}_j'ab\mathbf{w}_j''$, whence $\mathbf{u}'=s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}''=\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Since $\mathbf{J}_k^m\subseteq\mathbf{J}_r^m$ and $\mathbf{I}_k\subseteq\mathbf{I}_r$ by Lemma 6.4, we apply the statements proved in Subcases 1.1 and 1.2 to obtain the conclusion that $\mathbf{u}\approx\mathbf{u}'ba\mathbf{u}''$ holds in $\mathbf{V}$.

Subcase 2.2: $\mathbf{V}$ satisfies the hypothesis of (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u},a)\ne D(\mathbf{u},b)$. Subcase 2.1 allows us to assume that $a\notin\operatorname{con}(\mathbf{u}')$.

Suppose that $D(\mathbf{u},a)=s<r$. If $s\le k-1$ then a is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case because the first occurrence of a in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$. Therefore, $s\ge k$. Let (6.25) be the $(s-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0\le j\le n$ such that $\mathbf{w}_j=\mathbf{w}_j'ab\mathbf{w}_j''$, $\mathbf{u}'=s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}''=\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Put $\mathbf{u}^*=\mathbf{u}'ba\mathbf{u}''$. Since $a,b\notin\{s_1,s_2,\dots,s_n\}$, the $(s-1)$-decomposition of $\mathbf{u}^*$ has the form

$$s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j^*\cdots s_n\mathbf{w}_n$$

where $\mathbf{w}_j^*=\mathbf{w}_j'ba\mathbf{w}_j''$. Then (2.1) and (3.6) with $\mathbf{v}=\mathbf{u}^*$ and $\ell=s$ hold. Now Lemma 3.12 shows that $D(\mathbf{u}^*,a)=s$. Since $\mathbf{V}$ satisfies the identity β_s by Lemma 6.4, we apply the statement proved in Subcase 1.3 to deduce that the identity $\mathbf{u}^*=\mathbf{u}'ba\mathbf{u}''\approx\mathbf{u}'ab\mathbf{u}''=\mathbf{u}$ holds in $\mathbf{V}$.

Suppose now that $D(\mathbf{u},a)>r$. Let now (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0\le j\le n$ such that $\mathbf{w}_j=\mathbf{w}_j'ab\mathbf{w}_j''$, whence $\mathbf{u}'=s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}''=\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Since $\mathbf{H}_k\subseteq\mathbf{H}_r$ by Lemma 6.4, we apply the statement proved in Subcase 1.3 to obtain the identity $\mathbf{u}\approx\mathbf{u}'ba\mathbf{u}''$ in $\mathbf{V}$.

Subcase 2.3: $\mathbf{V}$ satisfies the hypothesis of claim (iv), i.e., α_k holds in $\mathbf{V}$. Subcase 2.2 allows us to assume that $D(\mathbf{u},a)=D(\mathbf{u},b)$. Put $D(\mathbf{u},a)=r$. Then the subword ab of $\mathbf{u}$ mentioned in the formulation of the lemma lies in some $(r-1)$-block of $\mathbf{u}$. Let (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists $0\le j\le n$ such that $\mathbf{w}_j=\mathbf{w}_j'ab\mathbf{w}_j''$, whence $\mathbf{u}'=s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}''=\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Since $\mathbf{F}_k\subseteq\mathbf{F}_r$ by Lemma 6.4, we apply the statement proved in Subcase 1.4 to obtain the identity $\mathbf{u}\approx\mathbf{u}'ba\mathbf{u}''$ in $\mathbf{V}$.

Case 3: $D(\mathbf{u},b)=\infty$. This case is also divided into three subcases.

Subcase 3.1: $\mathbf{V}$ satisfies the hypothesis of (i) or (ii). Let s be a non-negative integer. Repeating the arguments from Subcase 2.1, we find that the subword ab of $\mathbf{u}$ mentioned in the formulation of the lemma lies in some s-block of $\mathbf{u}$. By Remark 3.2, there is $r\ge k$ such that (6.25) is the ℓ-decomposition of $\mathbf{u}$ for any $\ell\ge r$. Then ab is a subword of $\mathbf{w}_j$ for some $0\le j\le n$. We have $\mathbf{w}_j=\mathbf{w}_j'ab\mathbf{w}_j''$ for some possibly empty words $\mathbf{w}_j'$ and $\mathbf{w}_j''$. Then $\mathbf{u}'=s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}''=\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. We now prove that

$$\operatorname{occ}_z(\mathbf{w}_j)\ge 2 \tag{6.26}$$

for any $z\in\operatorname{con}(\mathbf{w}_j)$. Suppose first that $s_j=h_1^r(\mathbf{u},z)$ and $\operatorname{occ}_z(\mathbf{w}_j)=1$. If $\operatorname{occ}_z(\mathbf{u})=1$ then z is a 0-divider of $\mathbf{u}$. Lemma 3.5(i) implies that then $z\in\{s_1,\dots,s_n\}$, a contradiction. Therefore, $\mathrm{occ}_z(\mathbf{u}) \ge 2$. Since $\mathrm{occ}_z(\mathbf{w}_j) = 1$, we have $s_j \ne h_2^r(\mathbf{u}, z)$. This means that $D(\mathbf{u}, z) \le r+1$. According to Lemma 3.7, z is an $(r+1)$-divider of $\mathbf{u}$, which contradicts the fact that (6.25) is the $(r+1)$-decomposition of $\mathbf{u}$. So, (6.26) is true whenever $s_j = h_1^r(\mathbf{u}, z)$. Suppose now that $s_j \ne h_1^r(\mathbf{u}, z)$. Then the $(1, r)$-restrictor of z in $\mathbf{u}$ is s_p for some $p < j$. This means that $z \in \mathrm{con}(s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_{j-1}\mathbf{w}_{j-1})$. Then

$$\mathbf{u} = \mathbf{f}z\mathbf{g}\mathbf{w}_{j1}z\mathbf{w}_{j2}s_{j+1}\mathbf{w}_{j+1} \cdots s_n\mathbf{w}_n$$

for some possibly empty words $\mathbf{f}, \mathbf{g}, \mathbf{w}_{j1}$ and $\mathbf{w}_{j2}$ with $\mathbf{f}z\mathbf{g} = s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_{j-1}\mathbf{w}_{j-1}s_j$ and $\mathbf{w}_j = \mathbf{w}_{j1}z\mathbf{w}_{j2}$. Then (4.9) implies that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{f}z\mathbf{g}\mathbf{w}_{j1}z^2\mathbf{w}_{j2}s_{j+1}\mathbf{w}_{j+1} \cdots s_n\mathbf{w}_n.$$

Therefore, we can assume that the claim (6.26) is true again. Thus, the claim holds for any $z \in \mathrm{con}(\mathbf{w}_j)$. Then Lemma 6.2(iii) implies that $\mathbf{V}$ satisfies $\mathbf{w}_j \approx \mathbf{w}_j'ba\mathbf{w}_j''$, whence

$$\begin{aligned}\mathbf{u} &= s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_j\mathbf{w}_js_{j+1}\mathbf{w}_{j+1} \cdots s_n\mathbf{w}_n\\ &\approx s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_j\mathbf{w}_j'ba\mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1} \cdots s_n\mathbf{w}_n = \mathbf{u}'ba\mathbf{u}''\end{aligned}$$

in this variety.

We have thus completed the proof of (i) and (ii).

Subcase 3.2: $\mathbf{V}$ satisfies the hypothesis of (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Then $D(\mathbf{u}, a) < \infty$. Put $D(\mathbf{u}, a) = r$. Repeating the arguments from Subcase 1.3, we have $a \notin \mathrm{con}(\mathbf{u}')$ and $r \ge k$. Let (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j'ab\mathbf{w}_j''$, $\mathbf{u}' = s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_j\mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1} \cdots s_n\mathbf{w}_n$. Put $\mathbf{u}^* = \mathbf{u}'ba\mathbf{u}''$. Since $a, b \notin \{s_1, \dots, s_n\}$, the $(r-1)$-decomposition of $\mathbf{u}^*$ has the form $s_0\mathbf{w}_0s_1\mathbf{w}_1 \cdots s_j\mathbf{w}_j^* \cdots s_n\mathbf{w}_n$ where $\mathbf{w}_j^* = \mathbf{w}_j'ba\mathbf{w}_j''$. Then (2.1) and (3.6) with $\mathbf{v} = \mathbf{u}^*$ and $\ell = r$ hold. Now Lemma 3.12 implies that $D(\mathbf{u}^*, a) = r$, whence a is an r-divider of $\mathbf{u}^*$ by Lemma 3.7. Then $h_1^r(\mathbf{u}^*, b) \ne h_2^r(\mathbf{u}^*, b)$. This implies that $D(\mathbf{u}^*, b) > r$. Since $\mathbf{V}$ satisfies the identity β_r by Lemma 6.4, we apply the statement proved in Subcase 1.3 to deduce that the identities $\mathbf{u}^* = \mathbf{u}'ba\mathbf{u}'' \approx \mathbf{u}'ab\mathbf{u}'' = \mathbf{u}$ hold in $\mathbf{V}$.

We have thus completed the proof of (iii).

Subcase 3.3: $\mathbf{V}$ satisfies the hypothesis of (iv), i.e., α_k holds in $\mathbf{V}$. Subcase 3.2 allows us to assume that $D(\mathbf{u}, a) = D(\mathbf{u}, b) = \infty$. This together with Lemma 3.7 implies that the subword ab of $\mathbf{u}$ mentioned in the formulation of the lemma lies in some s-block of $\mathbf{u}$ for any s. Now we repeat the arguments used in Subcase 3.1 and prove that $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$ holds in $\mathbf{V}$.

We have thus completed the proof of (iv) and of the entire lemma. ■

6.3. Reduction to intervals of the form $[\mathbf{F}_k, \mathbf{F}_{k+1}]$**.** Here we prove Proposition 6.1(3). We need several auxiliary results.

LEMMA 6.7. *Let* $\mathbf{V}$ *be a monoid variety such that* $\mathbf{V} \subseteq \mathbf{K}$ *and* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$*, and* s *be a natural number. Suppose that* (2.1) *and* (3.6) *with* $\ell = s$ *hold and there are letters* x *and* x_s *such that* $D(\mathbf{u}, x_s) = s$*,* $\ell_i(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$ *and* $\ell_1(\mathbf{v}, x_s) < \ell_i(\mathbf{v}, x)$ *for some* $i \in \{1, 2\}$*.*

(i) *If* $i = 1$ *then* $\mathbf{V} \subseteq \mathbf{H}_s$.
(ii) *If* $i = 2$ *then* $\mathbf{V} \subseteq \mathbf{J}_s^s$.

Proof. Lemma 6.2(iii) allows us to assume that $\mathrm{occ}_y(\mathbf{u}), \mathrm{occ}_y(\mathbf{v}) \le 2$ for any letter y. Now Lemma 3.14 implies that there are letters $x_0, x_1, \dots, x_{s-1}$ such that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$ for any $0 \le r < s$ and the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2s+1}$.

Suppose that $i = 1$. Then $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$ and $\ell_1(\mathbf{v}, x_s) < \ell_1(\mathbf{v}, x)$. Suppose that $\ell_1(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, x)$. In view of the above,

- the first occurrence of x in $\mathbf{u}$ lies in $\mathbf{u}_{2s+1}$,
- the second occurrence of x in $\mathbf{u}$ lies in $\mathbf{u}_{2s}\mathbf{u}_{2s-1} \cdots \mathbf{u}_0$,
- the first and second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_{2s}\mathbf{v}_{2s-1} \cdots \mathbf{v}_0$.

Now we substitute $x_s x^2$ for x_s in the identity $\mathbf{u} \approx \mathbf{v}$ to obtain the identity

$$\begin{aligned} &\mathbf{u}_{2s+1} x_s x^2 \mathbf{u}_{2s} x_{s-1} \mathbf{u}_{2s-1} x_s x^2 \mathbf{u}_{2s-2} x_{s-2} \mathbf{u}_{2s-3} x_{s-1} \cdots \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}_{2s+1} x_s x^2 \mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s x^2 \mathbf{v}_{2s-2} x_{s-2} \mathbf{v}_{2s-3} x_{s-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0. \end{aligned} \tag{6.27}$$

Further, we apply the identity (6.3) and delete the third and subsequent occurrences of x in both sides of (6.27) to obtain the identity

$$\begin{aligned} &\mathbf{u}_{2s+1} x_s x (\mathbf{u}_{2s} x_{s-1} \mathbf{u}_{2s-1} x_s \mathbf{u}_{2(s-1)} x_{s-2} \mathbf{u}_{2s-1} x_{s-1} \cdots \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0)_x \\ \approx\ &\mathbf{v}_{2s+1} x_s x^2 (\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s \mathbf{v}_{2(s-1)} x_{s-2} \mathbf{v}_{2s-1} x_{s-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0)_x. \end{aligned}$$

Now substitute 1 for all letters except $x, x_0, x_1, \dots, x_s$ to get

$$x x_s x x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx x_s x^2 x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1,$$

i.e., β_s. Therefore, $\mathbf{V} \subseteq \mathbf{H}_s$. Suppose now that $\ell_2(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$. In view of the above,

- the first and second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_{2s+1}$,
- the first and second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_{2s}\mathbf{v}_{2s-1} \cdots \mathbf{v}_0$.

Now we substitute $x_s x^2$ for x_s in the identity $\mathbf{u} \approx \mathbf{v}$ and obtain (6.27). The identity (6.3) allows us to delete the third and subsequent occurrences of x in both sides of (6.27). As a result, we obtain the identity

$$\begin{aligned} &\mathbf{u}_{2s+1} x_s \mathbf{u}_{2s} x_{s-1} \mathbf{u}_{2s-1} x_s \mathbf{u}_{2s-2} x_{s-2} \mathbf{u}_{2s-3} x_{s-1} \cdots \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}_{2s+1} x_s x^2 (\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s \mathbf{v}_{2(s-1)} x_{s-2} \mathbf{v}_{2s-1} x_{s-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0)_x. \end{aligned}$$

Now substitute 1 for all letters except $x, x_0, x_1, \dots, x_s$ to get

$$x^2 x_s x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx x_s x^2 x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1. \tag{6.28}$$

Then $\mathbf{V}$ satisfies the identities

$$\begin{aligned} x_s x^2 x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 &\stackrel{(6.28)}{\approx} x^2 x_s x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \\ &\stackrel{(4.5)}{\approx} x^3 x_s x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \\ &\stackrel{(6.28)}{\approx} x x_s x^2 x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \\ &\stackrel{(6.3)}{\approx} x x_s x x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1, \end{aligned}$$

whence β_s holds in $\mathbf{V}$. Therefore, $\mathbf{V} \subseteq \mathbf{H}_s$. Claim (i) is proved.

Suppose now that $i = 2$. Then $\ell_1(\mathbf{u}, x) < \ell_2(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$. If $\ell_1(\mathbf{v}, x_s) < \ell_1(\mathbf{v}, x)$ then we return to the already proved claim (i). So, we can assume that $\ell_1(\mathbf{v}, x) < \ell_1(\mathbf{v}, x_s)$. In view of the above,

- the first and second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_{2s+1}$,
- the first occurrence of x in $\mathbf{v}$ lies in $\mathbf{v}_{2s+1}$,
- the second occurrence of x in $\mathbf{v}$ lies in $\mathbf{v}_{2s}\mathbf{v}_{2s-1}\cdots\mathbf{v}_0$.

Now we substitute $x_s x^2$ for x_s in the identity $\mathbf{u} \approx \mathbf{v}$ and obtain (6.27). The identity (6.3) allows us to delete the third and subsequent occurrences of x in both sides of (6.27). As a result, we obtain the identity

$$\begin{aligned}&\mathbf{u}_{2s+1}x_s\mathbf{u}_{2s}x_{s-1}\mathbf{u}_{2s-1}x_s\mathbf{u}_{2s-2}x_{s-2}\mathbf{u}_{2s-3}x_{s-1}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0\\ &\approx \mathbf{v}_{2s+1}x_sx(\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}x_s\mathbf{v}_{2(s-1)}x_{s-2}\mathbf{v}_{2s-1}x_{s-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0)_x.\end{aligned}$$

Now substitute 1 for all letters except $x, x_0, x_1, \dots, x_s$ to get

$$x^2x_sx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1 \approx xx_sxx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1,$$

i.e., (6.4) with $k = s$. Lemma 6.3 implies now that $\mathbf{V} \subseteq \mathbf{J}_s^s$. Claim (ii) is proved. ■

Lemma 6.8. *Let* $\mathbf{V}$ *be a monoid variety such that* $\mathbf{V} \subseteq \mathbf{K}$ *and* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$. *If* (2.1) *holds, while* (3.6) *is false for some* $\ell > 1$, *then* $\mathbf{V} \subseteq \mathbf{J}_{\ell-1}^{\ell-1}$.

Proof. Suppose that (2.1) holds, while (3.6) is false for some $\ell = k > 1$, and k is the least number with this property. Then there exists a letter x such that $h_i^{k-1}(\mathbf{u}, x) \ne h_i^{k-1}(\mathbf{v}, x)$ where either $i = 1$ or $i = 2$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. In particular, the set of $(k-1)$-dividers of $\mathbf{u}$ is $\{t_0, t_1, \dots, t_m\}$. Since (3.6) with $\ell = k-1$ holds, Lemma 3.10 implies that $\mathbf{v}$ has the same $(k-1)$-dividers as $\mathbf{u}$ (but the order of the first occurrences of these letters in $\mathbf{u}$ and $\mathbf{v}$ may be different). Put $t_p = h_i^{k-1}(\mathbf{u}, x)$ and $t_q = h_i^{k-1}(\mathbf{v}, x)$. Clearly, $p \ne q$.

Suppose first that $\ell_i(\mathbf{u}, x) < \ell_1(\mathbf{u}, t_q)$. The choice of t_p and t_q guarantees that $\ell_1(\mathbf{u}, t_p) < \ell_i(\mathbf{u}, x)$ and $\ell_1(\mathbf{v}, t_q) < \ell_i(\mathbf{v}, x)$. Therefore, $\ell_1(\mathbf{u}, t_p) < \ell_1(\mathbf{u}, t_q)$, whence $p < q$ in this case. If t_q is simple in $\mathbf{u}$ then (2.1) implies that t_q is simple in $\mathbf{v}$ too. Therefore, t_q is a 0-divider of $\mathbf{u}$ and $\mathbf{v}$. Since $t_q = h_i^{k-1}(\mathbf{v}, x)$, we have $t_q = h_i^0(\mathbf{v}, x)$. Claim (3.6) with $\ell = 1$ implies that $t_q = h_i^0(\mathbf{u}, x)$. But this contradicts $p < q$. So, t_q is multiple in $\mathbf{u}$, whence t_q is multiple in $\mathbf{v}$ as well by (2.1). Therefore, $D(\mathbf{v}, t_q) > 0$. Moreover, $D(\mathbf{v}, t_q) \le k-1$ by Lemma 3.7 because t_q is a $(k-1)$-divider of $\mathbf{v}$. Put $r = D(\mathbf{v}, t_q)$. If $i = 1$ then Lemma 6.7(i) with $s = r$ and $x_s = t_q$ implies that $\mathbf{V} \subseteq \mathbf{H}_r \subseteq \mathbf{J}_{k-1}^{k-1}$. If $i = 2$ then $\mathbf{V} \subseteq \mathbf{J}_r^r \subseteq \mathbf{J}_{k-1}^{k-1}$ by Lemma 6.7(ii) with $s = r$ and $x_s = t_q$.

If $\ell_i(\mathbf{v}, x) < \ell_1(\mathbf{v}, t_p)$ then the argument is similar.

Finally, suppose that $\ell_1(\mathbf{u}, t_q) < \ell_i(\mathbf{u}, x)$ and $\ell_1(\mathbf{v}, t_p) < \ell_i(\mathbf{v}, x)$. The first of these inequalities implies that the first occurrence of t_q in $\mathbf{u}$ precedes the ith occurrence of x in $\mathbf{u}$. But t_p is the rightmost $(k-1)$-divider of $\mathbf{u}$ and precedes the ith occurrence of x. Therefore, $\ell_1(\mathbf{u}, t_q) < \ell_1(\mathbf{u}, t_p)$. Analogously, it follows from $\ell_1(\mathbf{v}, t_p) < \ell_i(\mathbf{v}, x)$ and $t_q = h_i^{k-1}(\mathbf{v}, x)$ that $\ell_1(\mathbf{v}, t_p) < \ell_1(\mathbf{v}, t_q)$. Suppose that t_p is simple in $\mathbf{u}$. Then (2.1) implies that t_p is simple in $\mathbf{v}$ too. Then t_p is a 0-divider of $\mathbf{u}$ and $\mathbf{v}$. Since $t_p = h_i^{k-1}(\mathbf{u}, x)$, we have $t_p = h_i^0(\mathbf{u}, x)$. Claim (3.6) with $\ell = 1$ implies that $t_p = h_i^0(\mathbf{v}, x)$. Note that

$\ell_1(\mathbf{v},t_p) < \ell_1(\mathbf{v},t_q) < \ell_i(\mathbf{v},x)$. Being the rightmost simple letter in $\mathbf{v}$ that is located to the left of x, the letter t_p is also the rightmost simple letter in $\mathbf{v}$ that is located to the left of t_q. In other words, $t_p = h_1^0(\mathbf{v},t_q)$. Claim (3.6) with $\ell = 1$ implies that $t_p = h_1^0(\mathbf{u},t_q)$. But this contradicts $\ell_1(\mathbf{u},t_q) < \ell_1(\mathbf{u},t_p)$. So, t_p is multiple in $\mathbf{u}$. Therefore, $D(\mathbf{u},t_p) > 0$. Moreover, $D(\mathbf{u},t_p) \le k-1$ by Lemma 3.7 because t_p is a $(k-1)$-divider of $\mathbf{u}$. Put $r = D(\mathbf{u},t_p)$. Then the hypothesis of Lemma 6.7 with $i = 1$, $s = r$, $x = t_q$ and $x_s = t_p$ holds. Therefore, Lemma 6.7(i) implies that $\mathbf{V} \subseteq \mathbf{H}_r \subseteq \mathbf{J}_{k-1}^{k-1}$. ∎

The following statement starts a series of similar assertions, which also includes Propositions 6.12, 6.14 and 6.17. These results provide solutions of the word problem in the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^m$ and $\mathbf{K}$. All of them are proved along similar lines. For the "only if" part, the scheme of proof is almost the same. As to the "if" part, the scheme is generally outlined in the proof of Proposition 6.9(i) but technically its implementation will get more and more complicated.

PROPOSITION 6.9. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds:*

(i) *in* $\mathbf{F}_k$ *if and only if* (2.1) *and* (3.6) *with* $\ell = k$ *hold;*
(ii) *in* $\mathbf{K}$ *if and only if* (2.1) *and* (3.6) *for all* ℓ *hold.*

Proof. (i) *Necessity.* Suppose that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{F}_k$. Proposition 2.2 and the inclusion $\mathbf{C}_2 \subseteq \mathbf{F}_k$ imply (2.1). Since $\mathbf{F}_k$ satisfies $\mathbf{u} \approx \mathbf{v}$, there is a sequence of words $\mathbf{u} = \mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n = \mathbf{v}$ such that, for any $i = 0,1,\dots,n-1$, there are words $\mathbf{p}_i, \mathbf{q}_i \in F^1$, an endomorphism ξ_i of F^1 and an identity $\mathbf{a}_i \approx \mathbf{b}_i$ from the system $\{\Phi,\alpha_k\}$ such that either $\mathbf{w}_i = \mathbf{p}_i\xi_i(\mathbf{a}_i)\mathbf{q}_i$ and $\mathbf{w}_{i+1} = \mathbf{p}_i\xi_i(\mathbf{b}_i)\mathbf{q}_i$, or $\mathbf{w}_i = \mathbf{p}_i\xi_i(\mathbf{b}_i)\mathbf{q}_i$ and $\mathbf{w}_{i+1} = \mathbf{p}_i\xi_i(\mathbf{a}_i)\mathbf{q}_i$. By induction we can assume without loss of generality that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi,\alpha_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \{xyx \approx xyx^2, x^2y \approx x^2yx\}$ then the assertion is obvious because the first and second occurrences of the letters of $\mathbf{u}$ do not take part in modifying $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with (4.4). Then, since $D(\mathbf{a},x) = D(\mathbf{a},y) = \infty$, Lemma 3.15 implies that the subword $\xi(\mathbf{a})$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{q}$ is contained in some s-block for all s. In particular, this subword is contained is some $(k-1)$-block. This implies (3.6) with $\ell = k$.

Finally, suppose that $\mathbf{a} \approx \mathbf{b}$ coincides with α_k. Then

$$\xi(\mathbf{a}) = \mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$

$$\xi(\mathbf{b}) = \mathbf{b}_k\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{b}_k$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$

$$\mathbf{v} = \mathbf{p}\mathbf{b}_k\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a},x_k) = D(\mathbf{a},y_k) = k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k\mathbf{b}_k$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{a}_{k-1}$ is contained in some $(k-1)$-block. This implies that (3.6) with $\ell = k$ is true.

Sufficiency. Let us outline the further argument; note that sufficiency in Propositions 6.12, 6.14 and 6.17 will be proved according to the same scheme. Let $\mathbf{u} \approx \mathbf{v}$ be an identity which satisfies the hypothesis of the proposition. We start by considering the $(k-1)$-decomposition of $\mathbf{u}$. Relying on Lemma 6.6 and using identities which hold in $\mathbf{F}_k$, we show that any $(k-1)$-block of $\mathbf{u}$ can be replaced by a word of some "canonical form". We replace all $(k-1)$-blocks of $\mathbf{u}$ in this way, getting some word $\mathbf{u}^\sharp$. Then we consider $\mathbf{v}$. It turns out that, up to identities in $\mathbf{F}_k$, this word has exactly the same $(k-1)$-blocks and $(k-1)$-dividers as $\mathbf{u}$. This allows us to change $(k-1)$-blocks of $\mathbf{v}$ in the same way as $(k-1)$-blocks of $\mathbf{u}$, getting the word $\mathbf{u}^\sharp$ again. This evidently implies that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{F}_k$.

Now we proceed to implementing the above plan. Suppose that $\mathbf{u} \approx \mathbf{v}$ satisfies (2.1) and (3.6) with $\ell = k$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. Fix $i \in \{0, 1, \dots, m\}$. Lemma 6.2(ii) allows us to suppose that every letter from $\mathrm{con}(\mathbf{u}_i)$ occurs in $\mathbf{u}_i$ at most twice. Put $\mathrm{mul}(\mathbf{u}_i) = \{x_1, \dots, x_p\}$, $\mathrm{sim}(\mathbf{u}_i) = \{y_1, \dots, y_q\}$ and

$$\overline{\mathbf{u}_i} = x_1^2 \cdots x_p^2 y_1 \cdots y_q.$$

Note that $\overline{\mathbf{u}_i}$ is nothing but the "canonical form" of the $(k-1)$-block $\mathbf{u}_i$ mentioned above. Indeed, $\mathbf{u} = \mathbf{w}_1 \mathbf{u}_i \mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Lemmas 6.2(ii) and 6.6(iv) imply now that $\mathbf{F}_k$ satisfies the identity

$$\mathbf{u} = \mathbf{w}_1 \mathbf{u}_i \mathbf{w}_2 \approx \mathbf{w}_1 \overline{\mathbf{u}_i} \mathbf{w}_2.$$

In particular, $\mathbf{F}_k$ satisfies the identities

$$\mathbf{u} = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \mathbf{u}_m \approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \overline{\mathbf{u}_m}.$$

Put $\mathbf{u}' = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \overline{\mathbf{u}_m}$. Note that (2.1) and (3.6) with $\mathbf{v} = \mathbf{u}'$ and $\ell = k$ hold. Then Lemma 3.8 implies that $\mathbf{u}$ and $\mathbf{u}'$ are $(k-1)$-equivalent, i.e., $t_0, t_1, \dots, t_m$ are $(k-1)$-dividers of $\mathbf{u}'$, while $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{m-1}, \overline{\mathbf{u}_m}$ are $(k-1)$-blocks of this word. Next, we can repeat the arguments above with $\mathbf{u}$ replaced by $\mathbf{u}'$ and obtain the identities

$$\mathbf{u}' = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \overline{\mathbf{u}_m} \approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \overline{\mathbf{u}_{m-1}} t_m \overline{\mathbf{u}_m}$$

in $\mathbf{F}_k$. Continuing, we find that $\mathbf{F}_k$ satisfies the identities

$$\begin{aligned} \mathbf{u} = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \mathbf{u}_m &\approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \overline{\mathbf{u}_m} \\ &\approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \overline{\mathbf{u}_{m-1}} t_m \overline{\mathbf{u}_m} \approx \cdots \approx t_0 \overline{\mathbf{u}_0} t_1 \overline{\mathbf{u}_1} \cdots t_m \overline{\mathbf{u}_m}. \end{aligned} \tag{6.29}$$

Put $\mathbf{u}^\sharp = t_0 \overline{\mathbf{u}_0} t_1 \overline{\mathbf{u}_1} \cdots t_m \overline{\mathbf{u}_m}$.

We now turn to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Claim (3.6) with $\ell = k$ implies that the jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if the jth occurrence of x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$, for any x and any $j = 1, 2$. We are going to check that $\mathrm{sim}(\mathbf{u}_i) = \mathrm{sim}(\mathbf{v}_i)$ and $\mathrm{mul}(\mathbf{u}_i) = \mathrm{mul}(\mathbf{v}_i)$. Let $x \in \mathrm{con}(\mathbf{u}_i)$. Lemma 6.2(ii) allows us to assume that $\mathrm{occ}_x(\mathbf{u}) \le 2$. There are three possibilities. First, if the first and second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_i$ then the first and second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_i$, whence $x \in \mathrm{mul}(\mathbf{u}_i)$ and $x \in \mathrm{mul}(\mathbf{v}_i)$. Second, if the first occurrence of x in $\mathbf{u}$ lies in $\mathbf{u}_i$ but the second does not, then the first occurrence of x in $\mathbf{v}$ lies in $\mathbf{v}_i$ but the second does not lie, whence $x \in \mathrm{sim}(\mathbf{u}_i)$ and $x \in \mathrm{sim}(\mathbf{v}_i)$. Finally, third, if the first occurrence of x in $\mathbf{u}$ is to the left of $\mathbf{u}_i$, while the second is in $\mathbf{u}_i$, then the first occurrence of x in $\mathbf{v}$ is to the left of $\mathbf{v}_i$, while the second is in $\mathbf{v}_i$. In this case we can apply the identity (4.9). This allows us to suppose that $x \in \mathrm{mul}(\mathbf{u}_i)$ and $x \in \mathrm{mul}(\mathbf{v}_i)$. Thus, $\mathrm{sim}(\mathbf{u}_i) = \mathrm{sim}(\mathbf{v}_i)$ and $\mathrm{mul}(\mathbf{u}_i) = \mathrm{mul}(\mathbf{v}_i)$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating the arguments above, we conclude that $\mathbf{F}_k$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$.

(ii) *Necessity* follows from (i) and the evident inclusion $\mathbf{F}_k \subseteq \mathbf{K}$, while *sufficiency* is proved in the same way as in (i). ∎

Now we are well prepared to quickly complete the proof of claim (3) of Proposition 6.1. Let $\mathbf{E} \subset \mathbf{X} \subset \mathbf{K}$. We have to verify that $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$ for some k. Suppose that $\mathbf{F}_1 \nsubseteq \mathbf{X}$. Then there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{F}_1$. Propositions 4.2 and 6.9(i) and the inclusion $\mathbf{E} \subseteq \mathbf{X}$ imply that (2.1) and (4.2) hold, while (3.6) with $\ell = 1$ is false. Let (3.4) be the 0-decomposition of $\mathbf{u}$. Then Lemma 3.8 implies that the 0-decomposition of $\mathbf{v}$ has the form (3.5). Since $\mathbf{u} \approx \mathbf{v}$ violates (3.6) with $\ell = 1$ but satisfies (4.2), there is a letter x such that $h_2^0(\mathbf{u}, x) \ne h_2^0(\mathbf{v}, x)$. Put $t_i = h_2^0(\mathbf{u}, x)$ and $t_j = h_2^0(\mathbf{v}, x)$. We may assume that $j < i$. Since (4.2) holds, we have $h_1^0(\mathbf{u}, x) = h_1^0(\mathbf{v}, x) = t_q$ for some q. Clearly, $q \le j$. Thus, the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\mathbf{u}_1 t_q \mathbf{u}_2 x \mathbf{u}_3 t_i \mathbf{u}_4 x \mathbf{u}_5 \approx \mathbf{v}_1 t_q \mathbf{v}_2 x \mathbf{v}_3 x \mathbf{v}_4 t_i \mathbf{v}_5$$

for some possibly empty words $\mathbf{u}_s$ and $\mathbf{v}_s$ with $s = 1, \dots, 5$. Substituting 1 for all letters in $\mathbf{u} \approx \mathbf{v}$ except x and t_i, we obtain an identity of the form $x t_i x^p \approx x^q t_i x^r$ where $p \ge 1$, $q \ge 2$ and $r \ge 0$. Now (6.3) implies that $\mathbf{X}$ satisfies $x t_i x \approx x^2 t_i$. This fact together with the inclusion $\mathbf{X} \subseteq \mathbf{K}$ shows that $\mathbf{X} \subseteq \mathbf{E}$, contradicting the choice of $\mathbf{X}$. Thus, $\mathbf{F}_1 \subseteq \mathbf{X}$. If $\mathbf{X}$ contains an infinite number of varieties of the form $\mathbf{F}_k$ then Proposition 6.9 implies that $\mathbf{X} = \mathbf{K}$. Hence there is a natural number k such that $\mathbf{F}_k \subseteq \mathbf{X}$ but $\mathbf{F}_{k+1} \nsubseteq \mathbf{X}$. Then Proposition 6.9(i) implies that (2.1) holds, while (3.6) with $\ell = k+1$ fails. Now we apply Lemma 6.8 to conclude that $\mathbf{X} \subseteq \mathbf{J}_k^k \subset \mathbf{F}_{k+1}$. Thus, $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$. Proposition 6.1(3) is proved.

6.4. Structure of the interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$. Here we prove Proposition 6.1(4). We divide this section into six subsections. In Subsections 6.4.1–6.4.5 we verify that each variety from the interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ coincides with one of $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^1$, $\mathbf{J}_k^2$, …, $\mathbf{J}_k^k$, $\mathbf{F}_{k+1}$. In Subsection 6.4.6 we check that all these varieties are pairwise different. These facts together with Lemma 6.4 imply Proposition 6.1(4).

6.4.1. If $\mathbf{F}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ then $\mathbf{H}_k \subseteq \mathbf{X}$. The first step in the verification of Proposition 6.1(4) is

LEMMA 6.10. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$ then either $\mathbf{X} = \mathbf{F}_k$ or $\mathbf{X} \supseteq \mathbf{H}_k$.*

To check this, we need several auxiliary results.

LEMMA 6.11. *Let $\mathbf{V}$ be a monoid variety with $\mathbf{F}_s \subseteq \mathbf{V} \subseteq \mathbf{K}$ for some s. If $\mathbf{V}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$ such that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$, $\ell_1(\mathbf{v}, b) < \ell_1(\mathbf{v}, a)$ and $D(\mathbf{u}, a) = D(\mathbf{u}, b) = s$ for some $a, b \in \mathrm{con}(\mathbf{u})$ then $\mathbf{V} = \mathbf{F}_s$.*

Proof. Put $x_s = a$ and $y_s = b$. Since $\mathbf{F}_s \subseteq \mathbf{V}$, Proposition 6.9(i) implies (2.1) and (3.6) with $\ell = s$. Suppose that

$$\ell_2(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, y_s) \quad \text{and} \quad \ell_2(\mathbf{v}, x_s) < \ell_2(\mathbf{v}, y_s). \tag{6.30}$$

Now Lemma 3.14 implies that there are letters $x_0, x_1, \dots, x_{s-1}$ such that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$ for any $0 \le r < s$ and the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2s+1}$.

One can verify that the first occurrences of x_s and y_s in $\mathbf{u}$ lie in the same $(s-1)$-block. Put $t_1 = h_1^{s-1}(\mathbf{u}, x_s)$ and $t_2 = h_1^{s-1}(\mathbf{u}, y_s)$. For a contradiction, suppose that $t_1 \ne t_2$. Since $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, y_s)$, we have $\ell_1(\mathbf{u}, t_1) < \ell_1(\mathbf{u}, t_2)$. Lemma 3.8 with $k = s - 1$ implies that $\ell_1(\mathbf{v}, t_1) < \ell_1(\mathbf{v}, t_2)$. In view of (3.6) with $\ell = s$, $t_1 = h_1^{s-1}(\mathbf{v}, x_s)$ and $t_2 = h_1^{s-1}(\mathbf{v}, y_s)$. But this contradicts $\ell_1(\mathbf{v}, y_s) < \ell_1(\mathbf{v}, x_s)$. So, the first occurrences of x_s and y_s in $\mathbf{u}$ lie in the same $(s-1)$-block. In particular, the first occurrence of y_s in $\mathbf{u}$ precedes the first occurrence of x_{s-1} in $\mathbf{u}$ because $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-1})$ and x_{s-1} is an $(s-1)$-divider. This implies that $\mathbf{u}_{2s} = \mathbf{u}_{2s}' y_s \mathbf{u}_{2s}''$ for some possibly empty words $\mathbf{u}_{2s}'$ and $\mathbf{u}_{2s}''$. Since the first occurrence of y_s in $\mathbf{v}$ precedes the first occurrence of x_s in $\mathbf{v}$, we have $\mathbf{v}_{2s+1} = \mathbf{v}_{2s+1}' y_s \mathbf{v}_{2s+1}''$ for some possibly empty words $\mathbf{v}_{2s+1}'$ and $\mathbf{v}_{2s+1}''$.

Further, since $\ell_1(\mathbf{u}, y_s) < \ell_1(\mathbf{u}, x_{s-2})$, we apply Lemma 3.13 with $\mathbf{w} = \mathbf{u}$, $z = y_s$, $t = x_{s-2}$ and $r = s$ to obtain $\ell_2(\mathbf{u}, y_s) < \ell_1(\mathbf{u}, x_{s-2})$. This implies that $\mathbf{u}_{2s-2} = \mathbf{u}_{2s-2}' y_s \mathbf{u}_{2s-2}''$ for some possibly empty words $\mathbf{u}_{2s-2}'$ and $\mathbf{u}_{2s-2}''$. Analogously, we can verify that $\mathbf{v}_{2s-2} = \mathbf{v}_{2s-2}' y_s \mathbf{v}_{2s-2}''$ for some possibly empty words $\mathbf{v}_{2s-2}'$ and $\mathbf{v}_{2s-2}''$.

In view of the above, the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned}
&\mathbf{u}_{2s+1} x_s \mathbf{u}_{2s}' \overset{(1)}{y_s} \mathbf{u}_{2s}' x_{s-1} \mathbf{u}_{2s-1} x_s \mathbf{u}_{2s-2}' \overset{(2)}{y_s} \mathbf{u}_{2s-2}' x_{s-2} \mathbf{u}_{2s-3} x_{s-1} \cdots \\
&\quad \cdot \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\
\approx\ &\mathbf{v}_{2s+1}' \overset{(1)}{y_s} \mathbf{v}_{2s+1}'' x_s \mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s \mathbf{v}_{2s-2}' \overset{(2)}{y_s} \mathbf{v}_{2s-2}'' x_{s-2} \mathbf{v}_{2s-3} x_{s-1} \cdots \\
&\quad \cdot \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.
\end{aligned}$$

Lemma 6.2(ii) allows us to assume that the letters x_r with $1 \le r \le s$ and y_s occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. Now substituting 1 for all letters except $x_0, x_1, \dots, x_s$ and y_s, we get the identity

$$x_s y_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx y_s x_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1,$$

i.e., α_s.

Suppose now that (6.30) is false. If $\ell_2(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, y_s)$ but $\ell_2(\mathbf{v}, y_s) < \ell_2(\mathbf{v}, x_s)$ then the same considerations as above show that $\mathbf{V}$ satisfies the identity

$$x_s y_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx \overset{(1)}{y_s} \overset{(1)}{x_s} x_{s-1} \overset{(2)}{y_s} \overset{(2)}{x_s} x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1.$$

According to Lemma 6.2(i), the variety $\mathbf{V}$ satisfies the identity σ_2. This allows us to transpose the second occurrences of x_s and y_s in the right-hand side of the last identity. As a result, we get α_s as well.

Finally, if $\ell_2(\mathbf{u}, y_s) < \ell_2(\mathbf{u}, x_s)$ then we can repeat the above arguments but apply Lemmas 3.13 and 3.14 for y_s rather than x_s. As a result, we obtain an identity

$$x_s y_s x_{s-1} y_s x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx y_s x_s x_{s-1} \mathbf{a} x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1$$

where

$$\mathbf{a} = \begin{cases} x_s y_s & \text{whenever } \ell_2(\mathbf{v}, x_s) < \ell_2(\mathbf{v}, y_s), \\ y_s x_s & \text{otherwise.} \end{cases}$$

If $\mathbf{a} = x_s y_s$, this identity coincides with α_s; otherwise we apply σ_2 once again and obtain α_s too. Thus, $\mathbf{V}$ satisfies α_s in any case, whence $\mathbf{V} \subseteq \mathbf{F}_s$. ∎

Proposition 6.12. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{H}_k$ *if and only if* (2.1), (3.6) *and*

$$\text{if either } D(\mathbf{u},x) \le \ell \text{ or } D(\mathbf{v},x) \le \ell \text{ then } h_1^\ell(\mathbf{u},x) = h_1^\ell(\mathbf{v},x) \tag{6.31}$$

with $\ell = k$ *all hold.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{H}_k$. Proposition 6.9(i) and the inclusion $\mathbf{F}_k \subseteq \mathbf{H}_k$ imply (2.1) and (3.6) with $\ell = k$. As in the proof of necessity in Proposition 6.9(i), we can assume that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \beta_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \Phi$ then (3.6) holds for any ℓ by Proposition 6.9(ii). Evidently, this implies the conclusion. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with β_k. Then

$$\xi(\mathbf{a}) = \mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$
$$\xi(\mathbf{b}) = \mathbf{a}_k\mathbf{a}_{k+1}^2\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{a}_{k+1}$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$
$$\mathbf{v} = \mathbf{p}\mathbf{a}_k\mathbf{a}_{k+1}^2\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a},x), D(\mathbf{a},x_k) > k-1$. Then Lemma 3.15 implies that the subword $\mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{a}_{k-1}$ is contained in some $(k-1)$-block. Moreover, in view of Lemma 3.15, no occurrence of the word $\mathbf{a}_{k+1}$ in $\mathbf{u}$ contains any k-dividers of $\mathbf{u}$ because $D(\mathbf{u},x) > k$ by Lemma 6.5. This means that $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent. Now Lemma 3.8 implies (6.31) with $\ell = k$.

Sufficiency. The outline of our argument here is the same as in the proof of sufficiency in Proposition 6.9(i), but the canonical form of a $(k-1)$-block of $\mathbf{u}$ is more complicated.

Suppose that (2.1), (3.6) and (6.31) with $\ell = k$ hold. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. Fix $i \in \{0, 1, \dots, m\}$. Let

$$t_i\mathbf{u}_i = s_0\mathbf{a}_0 s_1\mathbf{a}_1 \cdots s_n\mathbf{a}_n \tag{6.32}$$

be the presentation of $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n$. Put $\mathbf{u}_i^* = \mathbf{a}_0\mathbf{a}_1\cdots\mathbf{a}_n$. Let $\mathrm{con}(\mathbf{u}_i^*) = \{x_1, \dots, x_p\}$ and

$$\overline{\mathbf{u}_i} = x_1^2 \cdots x_p^2 s_1 \cdots s_n.$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the above mentioned "canonical form" of the $(k-1)$-block $\mathbf{u}_i$.

Clearly, $\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Suppose that $x \in \mathrm{con}(\mathbf{u}_i^*)$ but $x \notin \mathrm{con}(\mathbf{w}_1)$. If x is simple in $\mathbf{u}_i$ then x is a k-divider of $\mathbf{u}$, but this is not the case. Therefore, x is multiple in $\mathbf{u}_i$. Since $x \notin \mathrm{con}(\mathbf{w}_1)$, this means that the first and second occurrences of x in $\mathbf{u}$ lie in the same $(k-1)$-block of $\mathbf{u}$, whence $D(\mathbf{u},x) > k$. Further, Lemma 3.7 implies that $D(\mathbf{u},s_j) = k$ for all $j = 1, \dots, n$. We see that if $a \in \mathrm{con}(\mathbf{u}_i^*)$ and $b \in \{s_1, \dots, s_n\}$ then either $a \in \mathrm{con}(\mathbf{w}_1)$ or $D(\mathbf{u},a) \ne D(\mathbf{u},b)$. Now Lemma 6.6(ii)&(iii)

implies that the identities

$$\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2 \approx \mathbf{w}_1\mathbf{u}_i^* s_1 s_2 \cdots s_n \mathbf{w}_2$$

hold in $\mathbf{H}_k$. As we have seen above, if $x \in \mathrm{con}(\mathbf{u}_i^*) \setminus \mathrm{con}(\mathbf{w}_1)$ then $\mathrm{occ}_x(\mathbf{u}_i^*) \ge 2$. Further, if $x \in \mathrm{con}(\mathbf{w}_1) \cap \mathrm{con}(\mathbf{u}_i^*)$ then we can apply (4.9) to obtain $\mathrm{occ}_x(\mathbf{u}_i^*) \ge 2$ too. Now Lemma 6.2(ii) shows that $\mathrm{occ}_x(\mathbf{u}_i^*) = 2$ for any $x \in \mathrm{con}(\mathbf{u}_i^*)$. Then by Lemma 6.2(iii) the identities

$$\mathbf{u} \approx \mathbf{w}_1\mathbf{u}_i^* s_1 s_2 \cdots s_n \mathbf{w}_2 \approx \mathbf{w}_1\, \overline{\mathbf{u}_i}\, \mathbf{w}_2$$

hold in $\mathbf{H}_k$.

So, as in the proof of Proposition 6.9(i), using identities which hold in $\mathbf{H}_k$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one by one, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i = m, m-1, \dots, 0$. Thus $\mathbf{H}_k$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp = t_0\, \overline{\mathbf{u}_0}\, t_1\, \overline{\mathbf{u}_1} \cdots t_m\, \overline{\mathbf{u}_m}$.

We now turn to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). By (6.31) and Lemma 3.8, $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent. This means that $t_i\mathbf{v}_i$ is the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n$, i.e.,

$$t_i\mathbf{v}_i = s_0\mathbf{b}_0 s_1\mathbf{b}_1 \cdots s_n\mathbf{b}_n. \tag{6.33}$$

Claim (3.6) with $\ell = k$ implies that jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if the jth occurrence of a letter x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$, for any x and any $j = 1, 2$. Also, Lemma 6.2(ii) allows us to assume that if the first and second occurrences of x in $\mathbf{u}$ are both outside the $(k-1)$-block $\mathbf{u}_i$ then this letter does not occur in $\mathbf{u}_i$. Then $\mathrm{con}(\mathbf{u}_i^*) = \mathrm{con}(\mathbf{b}_0\mathbf{b}_1 \cdots \mathbf{b}_n)$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating the arguments given above, we conclude that $\mathbf{H}_k$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$. ■

Now we can complete the proof of Lemma 6.10. Let $\mathbf{F}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$. We have to verify that $\mathbf{X} \supseteq \mathbf{H}_k$. Suppose that $\mathbf{H}_k \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{H}_k$. Propositions 6.9(i) and 6.12 and the inclusion $\mathbf{F}_k \subset \mathbf{X}$ imply that (2.1) and (3.6) hold, while (6.31) with $\ell = k$ is false. According to Lemma 3.10, $\mathbf{u}$ and $\mathbf{v}$ have the same set of k-dividers but $\mathbf{u}$ and $\mathbf{v}$ are not k-equivalent by Lemma 3.8. Thus there are k-dividers a, b of $\mathbf{u}$, $\mathbf{v}$ such that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$, while $\ell_1(\mathbf{v}, b) < \ell_1(\mathbf{v}, a)$. In view of Lemma 3.7, $D(\mathbf{u}, a), D(\mathbf{u}, b) \le k$. Suppose that $D(\mathbf{u}, a) = r < k$. According to Lemma 3.11, claim (3.6) with $\ell = r$ holds. Then Lemma 3.12 implies that $D(\mathbf{v}, a) = r$. Also $\mathbf{u}$ and $\mathbf{v}$ are r-equivalent by Lemma 3.8. Put $c = h_1^r(\mathbf{u}, b)$. Since a is an r-divider of $\mathbf{u}$ by Lemma 3.7, the first occurrence of a in $\mathbf{u}$ precedes the first occurrence of c in $\mathbf{u}$. On the other hand, (3.6) with $\ell = r$ implies that $c = h_1^r(\mathbf{v}, b)$, whence $\ell_1(\mathbf{v}, c) < \ell_1(\mathbf{v}, a)$. This contradicts $\mathbf{u}$ and $\mathbf{v}$ being r-equivalent. So, $D(\mathbf{u}, a) = k$. Analogously, $D(\mathbf{u}, b) = k$. Now Lemma 6.11 with $s = k$ implies that $\mathbf{X} \subseteq \mathbf{F}_k$, a contradiction. Lemma 6.10 is proved. ■

6.4.2. If $\mathbf{H}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ then $\mathbf{I}_k \subseteq \mathbf{X}$. The second step in the verification of Proposition 6.1(4) is

LEMMA 6.13. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{H}_k, \mathbf{F}_{k+1}]$ then either $\mathbf{X} = \mathbf{H}_k$ or $\mathbf{X} \supseteq \mathbf{I}_k$.*

To check this, we need the following auxiliary result.

PROPOSITION 6.14. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in* $\mathbf{I}_k$ *if and only if* (2.1), (3.6) *and*

$$h_1^\ell(\mathbf{u}, x) = h_1^\ell(\mathbf{v}, x) \quad \text{for all } x \in \mathrm{con}(\mathbf{u}) \tag{6.34}$$

with $\ell = k$ *all hold.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{I}_k$. Proposition 6.12 and the inclusion $\mathbf{H}_k \subseteq \mathbf{I}_k$ imply that (2.1) and (3.6) with $\ell = k$ hold. As in the proof of Proposition 6.9(i), we can assume that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \gamma_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \Phi$ then (3.6) holds for any ℓ by Proposition 6.9(ii). Evidently, this implies the conclusion. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with γ_k. Then

$$\xi(\mathbf{a}) = \mathbf{b}_1\mathbf{b}_0\mathbf{b}_1\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$

$$\xi(\mathbf{b}) = \mathbf{b}_1\mathbf{b}_0\mathbf{a}_k\mathbf{b}_1\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{b}_0, \mathbf{b}_1$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{b}_1\mathbf{b}_0\mathbf{b}_1\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$

$$\mathbf{v} = \mathbf{p}\mathbf{b}_1\mathbf{b}_0\mathbf{a}_k\mathbf{b}_1\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a}, x_k) = k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_1$ and $\mathbf{a}_{k-1}$ does not contain any $(k-1)$-divider. Also, obviously, the subword $\mathbf{b}_1$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_k$ does not contain any s-divider, for any s. Therefore, the subword $\mathbf{b}_1\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_{k-1}$ lies in some $(k-1)$-block. It is evident that the subword $\mathbf{b}_1$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_k$ does not contain the first occurrence of any letter in $\mathbf{u}$. This implies that (6.34) with $\ell = k$ holds.

Sufficiency. As in the proof of Proposition 6.12, the outline of the argument here is similar to the proof of sufficiency in Proposition 6.9(i), but the canonical form of the block is even more complicated than in Proposition 6.12.

Suppose that (2.1), (3.6) and (6.34) with $\ell = k$ hold. As in the proof of sufficiency in Proposition 6.12, we suppose that (3.4) is the $(k-1)$-decomposition of $\mathbf{u}$, while (6.32) is the representation of $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n$.

For any $j = 0, 1, \dots, n$, we put

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid \text{the first occurrence of } x \text{ in } \mathbf{u} \text{ lies in } \mathbf{a}_j\}.$$

Note that X_j may be defined in another (equivalent) way: it is clear that a letter x lies in X_j if and only if it occurs in the k-block $\mathbf{a}_j$ and the $(1,k)$-restrictor of x in $\mathbf{u}$ coincides with the k-divider of $\mathbf{u}$ that immediately precedes $\mathbf{a}_j$. In other words,

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid s_j = h_1^k(\mathbf{u}, x)\}.$$

Put $X = X_0 \cup X_1 \cup \cdots \cup X_n$, $\mathbf{a}_j' = (\mathbf{a}_j)_X$ and $\mathbf{u}_i^* = \mathbf{a}_0'\mathbf{a}_1' \cdots \mathbf{a}_n'$. Let $X_j = \{x_{j1}, \dots, x_{jq_j}\}$, $\mathrm{con}(\mathbf{u}_i^*) = \{c_1, \dots, c_p\}$ and

$$\overline{\mathbf{u}_i} = (c_1 \cdots c_p) \cdot (x_{01}^2 \cdots x_{0q_0}^2) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2) \cdot (s_2 x_{21}^2 \cdots x_{2q_2}^2) \cdots (s_n x_{n1}^2 \cdots x_{nq_n}^2).$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the "canonical form" of the $(k-1)$-block $\mathbf{u}_i$ mentioned above.

Clearly, $\mathbf{u}=\mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Let $x\in X_j$. If x is simple in $\mathbf{u}_i$ then either x coincides with one of the k-dividers $s_1,\dots,s_n$ or $x\in\mathrm{con}(\mathbf{w}_1)$. But both cases contradict the choice of x. Therefore, x is multiple in $\mathbf{u}_i$. In view of (6.3), we can assume that $\mathrm{occ}_x(\mathbf{u}_i)=2$. Thus, $\mathbf{u}=\mathbf{a}x\mathbf{b}x\mathbf{c}$ for possibly empty words $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $x\mathbf{b}x$ is a subword of $\mathbf{u}_i$. We now verify that the variety $\mathbf{I}_k$ satisfies the identity $\mathbf{u}\approx\mathbf{a}x^2\mathbf{b}\mathbf{c}$. If $\mathbf{b}=\lambda$ then this is evident. Let now $\mathbf{b}\ne\lambda$. Then we apply Lemma 6.6(ii) and successively transpose the second occurrence of x in $\mathbf{u}$ with all the letters of the word $\mathbf{b}$ from right to left. Thus, $\mathbf{I}_k$ satisfies the identity $\mathbf{u}\approx\mathbf{a}x^2\mathbf{b}\mathbf{c}$. We can assume that $\ell_1(\mathbf{u},x_{j0})<\ell_1(\mathbf{u},x_{j1})<\cdots<\ell_1(\mathbf{u},x_{jq_j})$. Therefore, $\mathbf{I}_k$ satisfies the identity

$$\mathbf{u}\approx\mathbf{w}_1\cdot(x_{01}^2\cdots x_{0q_0}^2\mathbf{a}_0')\cdot(s_1x_{11}^2\cdots x_{1q_1}^2\mathbf{a}_1')\cdots(s_nx_{n1}^2\cdots x_{nq_n}^2\mathbf{a}_n')\cdot\mathbf{w}_2. \tag{6.35}$$

The definition of the set X and words of the form $\mathbf{a}_j'$ imply that $x\in\mathrm{con}(\mathbf{w}_1)$ for any $x\in\mathrm{con}(\mathbf{u}_i^*)$. Now we can apply Lemma 6.6(ii) to deduce that the identity

$$\mathbf{u}\approx\mathbf{w}_1\cdot\mathbf{u}_i^*\cdot(x_{01}^2\cdots x_{0q_0}^2)\cdot(s_1x_{11}^2\cdots x_{1q_1}^2)\cdots(s_nx_{n1}^2\cdots x_{nq_n}^2)\cdot\mathbf{w}_2$$

holds in $\mathbf{I}_k$. As seen above, $\mathrm{con}(\mathbf{u}_i^*)\subseteq\mathrm{con}(\mathbf{w}_1)$. Then we can apply (6.3) to deduce that the word $\mathbf{u}_i^*$ is linear. Then Lemma 6.2(i) implies that $\mathbf{I}_k$ satisfies the identities

$$\mathbf{u}\approx\mathbf{w}_1\cdot(c_1\cdots c_p)\cdot(x_{01}^2\cdots x_{0q_0}^2)\cdot(s_1x_{11}^2\cdots x_{1q_1}^2)\cdots(s_nx_{n1}^2\cdots x_{nq_n}^2)\cdot\mathbf{w}_2=\mathbf{w}_1\overline{\mathbf{u}_i}\,\mathbf{w}_2.$$

So, as in the proof of Proposition 6.9(i), using identities which hold in $\mathbf{I}_k$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one by one, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i=m,m-1,\dots,0$. Thus $\mathbf{I}_k$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp=t_0\,\overline{\mathbf{u}_0}\,t_1\,\overline{\mathbf{u}_1}\,\cdots t_m\,\overline{\mathbf{u}_m}$.

We now return to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Furthermore, (6.34) with $\ell=k$ and Lemma 3.8 imply that (6.33) is a representation of $t_i\mathbf{v}_i$ as the product of alternating k-dividers $s_0,s_1,\dots,s_n$ and k-blocks $\mathbf{b}_0,\mathbf{b}_1,\dots,\mathbf{b}_n$. Claim (6.34) implies that

$$X_j=\{x\in\mathrm{con}(\mathbf{b}_j)\mid s_j=h_1^k(\mathbf{v},x)\}$$

for all $j=0,1,\dots,r_i$. Put $\mathbf{b}_j'=(\mathbf{b}_j)_X$. Claim (3.6) with $\ell=k$ implies that the jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if the jth occurrence of x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$, for any x and any $j=1,2$. Also, Lemma 6.2(ii) allows us to assume that if the first and second occurrences of x in $\mathbf{u}$ do not lie in the $(k-1)$-block $\mathbf{u}_i$ then x does not occur in $\mathbf{u}_i$. Thus $\mathrm{con}(\mathbf{u}_i^*)=\mathrm{con}(\mathbf{b}_0'\mathbf{b}_1'\cdots\mathbf{b}_n')$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating the arguments above, we find that $\mathbf{I}_k$ satisfies the identities $\mathbf{v}\approx\mathbf{u}^\sharp\approx\mathbf{u}$. ∎

Now we can complete the proof of Lemma 6.13. Let $\mathbf{H}_k\subset\mathbf{X}\subseteq\mathbf{F}_{k+1}$. We have to verify that $\mathbf{X}\supseteq\mathbf{I}_k$. Suppose that $\mathbf{I}_k\nsubseteq\mathbf{X}$. Then there exists an identity $\mathbf{u}\approx\mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{I}_k$. Then Propositions 6.12 and 6.14 and the inclusion $\mathbf{H}_k\subset\mathbf{X}$ imply that (2.1), (3.6) and (6.31) with $\ell=k$ are true, while (6.34) with $\ell=k$ is false. Let (3.4) be the k-decomposition of $\mathbf{u}$. Claim (6.31) and Lemma 3.8 imply that the k-decomposition of $\mathbf{v}$ has the form (3.5). Since (6.34) is false, there is a letter x such that $h_1^k(\mathbf{u},x)\ne h_1^k(\mathbf{v},x)$. Put $t_i=h_1^k(\mathbf{u},x)$ and $t_j=h_1^k(\mathbf{v},x)$. Then $i\ne j$. We can assume that $i<j$. Then $\ell_1(\mathbf{u},x)<\ell_1(\mathbf{u},t_j)$, while $\ell_1(\mathbf{v},t_j)<\ell_1(\mathbf{u},x)$. Lemma 3.7 implies that $D(\mathbf{u},t_j)\le k$. Put $D(\mathbf{u},t_j)=r$. If $r=0$ then t_j is a 0-divider of $\mathbf{u}$. Claim (2.1) implies that t_j is a 0-divider of $\mathbf{v}$ too. Then $t_j = h_1^0(\mathbf{v}, x)$ but $t_j \ne h_1^0(\mathbf{u}, x)$. In view of Lemma 3.11, claim (3.6) with $\ell = p$ holds for any $1 \le p \le k$, a contradiction. Thus, $r \ge 1$. Now Lemma 6.7(i) with $s = r$ and $x_s = t_j$ applies, and we conclude that $\mathbf{X} \subseteq \mathbf{H}_r \subseteq \mathbf{H}_k$, a contradiction. Lemma 6.13 is proved. ■

6.4.3. If $\mathbf{I}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ then $\mathbf{J}_k^1 \subseteq \mathbf{X}$. The third step in the verification of Proposition 6.1(4) is

Lemma 6.15. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{I}_k, \mathbf{F}_{k+1}]$ then either $\mathbf{X} = \mathbf{I}_k$ or $\mathbf{X} \supseteq \mathbf{J}_k^1$.*

To check this, we need

Lemma 6.16. *Let $\mathbf{V}$ be a monoid variety with $\mathbf{V} \subseteq \mathbf{K}$ and ℓ a natural number. Suppose that $\mathbf{V}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$.*

(i) *If (2.1), (3.6) and (6.34) hold but the claim*
$$\text{if } x \in \operatorname{con}(\mathbf{u}) \text{ and } D(\mathbf{u}, x) \le m \text{ then } h_2^\ell(\mathbf{u}, x) = h_2^\ell(\mathbf{v}, x) \tag{6.36}$$
with $m = 1$ is false then $\mathbf{V} \subseteq \mathbf{I}_\ell$.

(ii) *If (2.1), (3.6), (6.34) and (6.36) with $m = r$ hold but (6.36) with $m = r + 1$ is false then $\mathbf{V} \subseteq \mathbf{J}_\ell^r$.*

Proof. Proofs of (i) and (ii) have the same initial part. Suppose that $\mathbf{V}$ satisfies the hypothesis of one of these two claims. Then (2.1), (3.6) and (6.34) hold. Let m be least such that (6.36) is false. Then there is a letter y_m such that $D(\mathbf{u}, y_m) = m$ and $h_2^\ell(\mathbf{u}, y_m) \ne h_2^\ell(\mathbf{v}, y_m)$. Put $x_\ell = h_2^\ell(\mathbf{u}, y_m)$ and $z_\ell = h_2^\ell(\mathbf{v}, y_m)$. In view of Lemma 3.7, we have $D(\mathbf{u}, x_\ell), D(\mathbf{u}, z_\ell) \le \ell$. Note that either $D(\mathbf{u}, x_\ell) = \ell$ or $D(\mathbf{u}, z_\ell) = \ell$. Indeed, if $D(\mathbf{u}, x_\ell), D(\mathbf{u}, z_\ell) < \ell$ then $D(\mathbf{v}, x_\ell), D(\mathbf{v}, z_\ell) < \ell$ by Lemma 3.12. Then x_ℓ and z_ℓ are $(\ell-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$, whence $x_\ell = h_2^{\ell-1}(\mathbf{u}, y_m)$ and $z_\ell = h_2^{\ell-1}(\mathbf{v}, y_m)$. But this contradicts (3.6). Suppose without loss of generality that $D(\mathbf{u}, x_\ell) = \ell$. By symmetry, we may assume that the first occurrence of z_ℓ in $\mathbf{u}$ precedes the first occurrence of x_ℓ in this word. Since (6.34) holds, $\ell_1(\mathbf{v}, z_\ell) < \ell_1(\mathbf{v}, x_\ell)$ by Lemma 3.8. This implies that $\ell_2(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, x_\ell)$.

Now Lemma 3.14 with $x_s = x_\ell$ and $s = \ell$ shows that there are letters $x_0, x_1, \dots, x_{\ell-1}$ such that, for any $p = 0, 1, \dots, \ell-1$ and $q = 0, 1, \dots, \ell-2$, $D(\mathbf{u}, x_p) = D(\mathbf{v}, x_p) = p$ and the inequalities
$$\ell_1(\mathbf{w}, x_{p+1}) < \ell_1(\mathbf{w}, x_p) < \ell_2(\mathbf{w}, x_{p+1}) \quad \text{and} \quad \ell_2(\mathbf{w}, x_{q+2}) < \ell_1(\mathbf{w}, x_q)$$
hold for $\mathbf{w} = \mathbf{u}, \mathbf{v}$.

Put $y_{m-1} = h_2^{m-1}(\mathbf{u}, y_m)$. According to Lemma 3.9, $D(\mathbf{u}, y_{m-1}) = m-1$ and $\ell_1(\mathbf{u}, y_m) < \ell_1(\mathbf{u}, y_{m-1})$. Moreover, (3.6) and Lemma 3.11 imply that $h_2^{m-1}(\mathbf{v}, y_m) = h_2^{m-1}(\mathbf{u}, y_m) = y_{m-1}$. Now we apply Lemma 3.9 again to obtain $D(\mathbf{v}, y_{m-1}) = m-1$ and $\ell_1(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, y_{m-1})$. In view of Lemma 3.7, the letter $x_{\ell-1}$ is an ℓ-divider of $\mathbf{u}$, whence $\ell_2(\mathbf{u}, y_m) < \ell_1(\mathbf{u}, x_{\ell-1})$ because $x_\ell = h_2^\ell(\mathbf{u}, y_m)$ and $\ell_1(\mathbf{u}, x_\ell) < \ell_1(\mathbf{u}, x_{\ell-1})$.

Lemma 6.2(ii) allows us to assume that the letters y_m and x_p with $1 \le p \le \ell$ occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. Further considerations are divided into two cases corresponding to statements (i) and (ii).

Case 1: $m=1$. In view of the above, the identity $\mathbf{u}\approx\mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+4}y_1\mathbf{u}_{2\ell+3}y_0\mathbf{u}_{2\ell+2}x_\ell\mathbf{u}_{2\ell+1}y_1\mathbf{u}_{2\ell}x_{\ell-1}\mathbf{u}_{2\ell-1}x_\ell\mathbf{u}_{2\ell-2}x_{\ell-2}\mathbf{u}_{2\ell-3}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0\\\approx{}&\mathbf{v}_{2\ell+4}y_1\mathbf{v}_{2\ell+3}y_0\mathbf{v}_{2\ell+2}y_1\mathbf{v}_{2\ell+1}x_\ell\mathbf{v}_{2\ell}x_{\ell-1}\mathbf{v}_{2\ell-1}x_\ell\mathbf{v}_{2\ell-2}x_{\ell-2}\mathbf{v}_{2\ell-3}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_{2\ell+4}$ and $\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_{2\ell+4}$ such that $x_s,y_0,y_1\notin\mathrm{con}(\mathbf{u}_i\mathbf{v}_i)$ for $0\le s\le\ell$ and $0\le i\le 2\ell+4$. Now substituting 1 for all letters occurring in this identity except $x_0,x_1,\dots,x_\ell,y_0$ and y_1, we get the identity

$$y_1y_0x_\ell y_1x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_1x_2x_0x_1\approx y_1y_0y_1x_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_1x_2x_0x_1,$$

i.e., γ_ℓ. Claim (i) is proved.

Case 2: $m>1$. Now we will prove that $\ell_2(\mathbf{v},x_m)<\ell_2(\mathbf{v},y_{m-1})$ and $\ell_2(\mathbf{u},x_m)<\ell_2(\mathbf{u},y_{m-1})$. Put $y_{m-2}=h_2^{m-2}(\mathbf{v},y_{m-1})$. Since $D(\mathbf{v},y_{m-1})=m-1$, Lemma 3.9 implies that $D(\mathbf{v},y_{m-2})=m-2$ and $\ell_1(\mathbf{v},y_{m-1})<\ell_1(\mathbf{v},y_{m-2})$. Recall that $\ell_1(\mathbf{v},y_m)<\ell_1(\mathbf{v},y_{m-1})$, whence $\ell_1(\mathbf{v},y_m)<\ell_1(\mathbf{v},y_{m-2})$. Since $D(\mathbf{v},y_m)=m$, we can apply Lemma 3.13 to conclude that $\ell_2(\mathbf{v},y_m)<\ell_1(\mathbf{v},y_{m-2})$. The first occurrence of x_ℓ in $\mathbf{v}$ precedes the second occurrence of y_m, whence $\ell_1(\mathbf{v},x_\ell)<\ell_1(\mathbf{v},y_{m-2})$. Then Lemma 3.13 implies that $\ell_2(\mathbf{v},x_\ell)<\ell_1(\mathbf{v},y_{m-2})$. This yields $\ell_1(\mathbf{v},x_{\ell-1})<\ell_2(\mathbf{v},x_\ell)<\ell_1(\mathbf{v},y_{m-2})$. If $\ell-1\ge m$ then Lemma 3.13 shows that $\ell_2(\mathbf{v},x_{\ell-1})<\ell_1(\mathbf{v},y_{m-2})$. Continuing, we eventually obtain $\ell_2(\mathbf{v},x_m)<\ell_1(\mathbf{v},y_{m-2})$. In particular, $\ell_1(\mathbf{v},x_m)<\ell_1(\mathbf{v},y_{m-2})$. In view of Lemma 3.7, the letters x_m and y_{m-2} are ℓ-dividers of $\mathbf{v}$. Now Lemma 3.8 yields $\ell_1(\mathbf{u},x_m)<\ell_1(\mathbf{u},y_{m-2})$. Then Lemma 3.13 shows that $\ell_2(\mathbf{u},x_m)<\ell_1(\mathbf{u},y_{m-2})$. The choice of y_{m-2} implies that the first occurrence of y_{m-2} in $\mathbf{v}$ precedes the second occurrence of y_{m-1}. Therefore, $\ell_2(\mathbf{v},x_m)<\ell_2(\mathbf{v},y_{m-1})$. In view of (3.6) and Lemma 3.11, $h_2^{m-2}(\mathbf{u},y_{m-1})=h_2^{m-2}(\mathbf{v},y_{m-1})=y_{m-2}$, whence $\ell_2(\mathbf{u},x_m)<\ell_2(\mathbf{u},y_{m-1})$.

Let now $m>2$. Note that

$$\ell_1(\mathbf{u},y_{m-1})<\ell_2(\mathbf{u},y_m)<\ell_1(\mathbf{u},x_\ell)<\ell_1(\mathbf{u},x_{\ell-1})<\cdots<\ell_1(\mathbf{u},x_{m-3}).$$

If $\ell_1(\mathbf{u},x_{m-3})<\ell_2(\mathbf{u},y_{m-1})$ then x_{m-3} lies between the first and the second occurrences of y_{m-1} in $\mathbf{u}$. Since x_{m-3} is an $(m-3)$-divider of $\mathbf{u}$, we obtain a contradiction with the equality $D(\mathbf{u},y_{m-1})=m-1$. Thus, $\ell_2(\mathbf{u},y_{m-1})<\ell_1(\mathbf{u},x_{m-3})$ whenever $m>2$.

Further, there are three possibilities for the second occurrence of y_{m-1} in $\mathbf{u}$:

$$\ell_2(\mathbf{u},y_{m-1})<\ell_1(\mathbf{u},x_{m-2}); \tag{6.37}$$

$$\ell_1(\mathbf{u},x_{m-2})<\ell_2(\mathbf{u},y_{m-1})<\ell_2(\mathbf{u},x_{m-1}); \tag{6.38}$$

$$\ell_2(\mathbf{u},x_{m-1})<\ell_2(\mathbf{u},y_{m-1}). \tag{6.39}$$

Suppose that (6.37) holds. Then $\ell_1(\mathbf{u},y_{m-2})<\ell_1(\mathbf{u},x_{m-2})$. In view of Lemma 3.8, $\ell_1(\mathbf{v},y_{m-2})<\ell_1(\mathbf{v},x_{m-2})$. Since $y_{m-2}=h_2^{m-2}(\mathbf{v},y_{m-1})$ by Lemma 3.11, we have $\ell_2(\mathbf{v},y_{m-1})<\ell_1(\mathbf{v},x_{m-2})$. Now if $m>2$ then we apply Lemma 3.14 with $x_s=y_{m-2}$ and $s=m-2$ to conclude that there are letters $y_0,y_1,\dots,y_{m-3}$ such that $D(\mathbf{u},y_p)=D(\mathbf{v},y_p)=p$ and, for any $p=0,1,\dots,m-2$, the inequalities

$$\ell_1(\mathbf{w},y_{p+1})<\ell_1(\mathbf{w},y_p)<\ell_2(\mathbf{w},y_{p+1})\quad\text{and}\quad\ell_2(\mathbf{w},y_{p+2})<\ell_1(\mathbf{w},y_p)$$

hold for $\mathbf{w}=\mathbf{u},\mathbf{v}$. Lemma 6.2(ii) allows us to assume that the letters y_p with $1\le p\le m$ occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. In view of the above, the identity $\mathbf{u}\approx\mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+5}y_m\mathbf{u}_{2\ell+4}y_{m-1}\mathbf{u}_{2\ell+3}x_\ell\mathbf{u}_{2\ell+2}y_m\mathbf{u}_{2\ell+1}x_{\ell-1}\mathbf{u}_{2\ell}x_\ell\mathbf{u}_{2\ell-1}x_{\ell-2}\mathbf{u}_{2\ell-2}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{u}_{2m+1}y_{m-2}\mathbf{u}_{2m}y_{m-1}\mathbf{u}_{2m-1}x_{m-1}\mathbf{u}_{2m-2}y_{m-3}\mathbf{u}_{2m-2}y_{m-2}\cdots\mathbf{u}_4y_1\mathbf{u}_3y_2\mathbf{u}_2y_0\mathbf{u}_1y_1\mathbf{u}_0\\\approx\ &\mathbf{v}_{2\ell+5}y_m\mathbf{v}_{2\ell+4}y_{m-1}\mathbf{v}_{2\ell+3}y_m\mathbf{v}_{2\ell+2}x_\ell\mathbf{v}_{2\ell+1}x_{\ell-1}\mathbf{v}_{2\ell}x_\ell\mathbf{v}_{2\ell-1}x_{\ell-2}\mathbf{v}_{2\ell-2}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{v}_{2m+1}y_{m-2}\mathbf{v}_{2m}y_{m-1}\mathbf{v}_{2m-1}x_{m-1}\mathbf{v}_{2m-2}y_{m-3}\mathbf{v}_{2m-2}y_{m-2}\cdots\mathbf{v}_4y_1\mathbf{v}_3y_2\mathbf{v}_2y_0\mathbf{v}_1y_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_{2\ell+5}$ such that $x_s,y_t\notin\mathrm{con}(\mathbf{u}_i\mathbf{v}_i)$ for $m-1\le s\le\ell$, $0\le t\le m$ and $0\le i\le 2\ell+5$. Now substituting 1 for all letters occurring in this identity except $y_0,y_1,\dots,y_m$ and $x_{m-1},x_m,\dots,x_\ell$, we get the identity

$$\begin{aligned}&y_my_{m-1}x_\ell y_mx_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots y_{m-2}y_{m-1}x_{m-1}y_{m-3}y_{m-2}\cdots y_1y_2y_0y_1\\\approx\ &y_my_{m-1}y_mx_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots y_{m-2}y_{m-1}x_{m-1}y_{m-3}y_{m-2}\cdots y_1y_2y_0y_1.\end{aligned}$$

Now we rename y_i as x_i for $i=0,1,\dots,m-2$ to obtain the identity

$$\begin{aligned}&y_m\overset{(1)}{y_{m-1}}x_\ell y_mx_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}\overset{(2)}{y_{m-1}}\overset{(2)}{x_{m-1}}x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1\\\approx\ &y_m\overset{(1)}{y_{m-1}}y_mx_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}\overset{(2)}{y_{m-1}}\overset{(2)}{x_{m-1}}x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1.\end{aligned}\tag{6.40}$$

In view of Lemma 6.2(i), we may use the identity σ_2, which allows us to swap the second occurrences of x_{m-1} and y_{m-1} in both sides of (6.40). As a result, we get δ_ℓ^{m-1}.

Suppose now that (6.38) holds. If $\ell_2(\mathbf{v},y_{m-1})<\ell_1(\mathbf{v},x_{m-2})$ then $\ell_1(\mathbf{v},y_{m-2})<\ell_1(\mathbf{v},x_{m-2})$. In view of Lemma 3.8, $\ell_1(\mathbf{u},y_{m-2})<\ell_1(\mathbf{u},x_{m-2})$. Since $y_{m-2}=h_2^{m-2}(\mathbf{v},y_{m-1})$ by Lemma 3.11, we have $\ell_2(\mathbf{u},y_{m-1})<\ell_1(\mathbf{u},x_{m-2})$. This contradicts (6.38). Thus, $\ell_1(\mathbf{v},x_{m-2})<\ell_2(\mathbf{v},y_{m-1})$.

Suppose that $\ell_2(\mathbf{v},y_{m-1})<\ell_2(\mathbf{v},x_{m-1})$. Then the identity $\mathbf{u}\approx\mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+5}y_m\mathbf{u}_{2\ell+4}y_{m-1}\mathbf{u}_{2\ell+3}x_\ell\mathbf{u}_{2\ell+2}y_m\mathbf{u}_{2\ell+1}x_{\ell-1}\mathbf{u}_{2\ell}x_\ell\mathbf{u}_{2\ell-1}x_{\ell-2}\mathbf{u}_{2\ell-2}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{u}_{2m+1}x_{m-2}\mathbf{u}_{2m}y_{m-1}\mathbf{u}_{2m-1}x_{m-1}\mathbf{u}_{2m-2}x_{m-3}\mathbf{u}_{2m-2}x_{m-2}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0\\\approx\ &\mathbf{v}_{2\ell+5}y_m\mathbf{v}_{2\ell+4}y_{m-1}\mathbf{v}_{2\ell+3}y_m\mathbf{v}_{2\ell+2}x_\ell\mathbf{v}_{2\ell+1}x_{\ell-1}\mathbf{v}_{2\ell}x_\ell\mathbf{v}_{2\ell-1}x_{\ell-2}\mathbf{v}_{2\ell-2}x_{\ell-1}\cdots\\&\quad\cdot\mathbf{v}_{2m+1}x_{m-2}\mathbf{v}_{2m}y_{m-1}\mathbf{v}_{2m-1}x_{m-1}\mathbf{v}_{2m-2}x_{m-3}\mathbf{v}_{2m-2}x_{m-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_{2\ell+5}$ such that $x_s,y_{m-1},y_m\notin\mathrm{con}(\mathbf{u}_i\mathbf{v}_i)$ for $0\le s\le\ell$ and $0\le i\le 2\ell+5$. Now substituting 1 for all letters occurring in this identity except $y_{m-1},y_m,x_0,x_1,\dots,x_\ell$, we get (6.40). As above, combining this identity with σ_2, we get δ_ℓ^{m-1}.

If $\ell_2(\mathbf{v},x_{m-1})<\ell_2(\mathbf{v},y_{m-1})$ then the same arguments as above show that the identity

$$\begin{aligned}&y_m\overset{(1)}{y_{m-1}}x_\ell y_mx_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}\overset{(2)}{y_{m-1}}\overset{(2)}{x_{m-1}}x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1\\\approx\ &y_my_{m-1}y_mx_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}x_{m-1}y_{m-1}x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1\end{aligned}$$

holds in $\mathbf{V}$. Now we apply σ_2 to the left-hand side to get δ_ℓ^{m-1}.

Finally, suppose that (6.39) holds. Suppose that $\ell_2(\mathbf{v},y_{m-1})<\ell_2(\mathbf{v},x_{m-1})$. Then the identity $\mathbf{u}\approx\mathbf{v}$ has the form

$$\mathbf{u}_{2\ell+5}y_m\mathbf{u}_{2\ell+4}y_{m-1}\mathbf{u}_{2\ell+3}x_\ell\mathbf{u}_{2\ell+2}y_m\mathbf{u}_{2\ell+1}x_{\ell-1}\mathbf{u}_{2\ell}x_\ell\mathbf{u}_{2\ell-1}x_{\ell-2}\mathbf{u}_{2\ell-2}x_{\ell-1}\cdots$$
$$\cdot\, \mathbf{u}_{2m+1}x_{m-2}\mathbf{u}_{2m}x_{m-1}\mathbf{u}_{2m-1}y_{m-1}\mathbf{u}_{2m-2}x_{m-3}\mathbf{u}_{2m-2}x_{m-2}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0$$
$$\approx \mathbf{v}_{2\ell+5}y_m\mathbf{v}_{2\ell+4}y_{m-1}\mathbf{v}_{2\ell+3}y_m\mathbf{v}_{2\ell+2}x_\ell\mathbf{v}_{2\ell+1}x_{\ell-1}\mathbf{v}_{2\ell}x_\ell\mathbf{v}_{2\ell-1}x_{\ell-2}\mathbf{v}_{2\ell-2}x_{\ell-1}\cdots$$
$$\cdot\, \mathbf{v}_{2m+1}x_{m-2}\mathbf{v}_{2m}y_{m-1}\mathbf{v}_{2m-1}x_{m-1}\mathbf{v}_{2m-2}x_{m-3}\mathbf{v}_{2m-2}x_{m-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0$$

for some possibly empty words $\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_{2\ell+5}$ such that $x_s,y_{m-1},y_m\notin\operatorname{con}(\mathbf{u}_i\mathbf{v}_i)$ for $0\le s\le\ell$ and $0\le i\le 2\ell+5$. Now substituting 1 for all letters occurring in this identity except $y_{m-1},y_m,x_0,x_1,\dots,x_\ell$, we get the identity

$$y_my_{m-1}x_\ell y_mx_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}x_{m-1}y_{m-1}x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1$$
$$\approx y_m\,\overset{(1)}{y_{m-1}}\,y_mx_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_{m-2}\,\overset{(2)}{y_{m-1}}\overset{(2)}{x_{m-1}}\,x_{m-3}x_{m-2}\cdots x_1x_2x_0x_1.$$

Applying once again σ_2 to the right-hand side, we get δ_ℓ^{m-1}.

If $\ell_2(\mathbf{v},x_{m-1})<\ell_2(\mathbf{v},y_{m-1})$ then the same arguments show that δ_ℓ^{m-1} holds in $\mathbf{V}$. ∎

Proposition 6.17. *A non-trivial identity $\mathbf{u}\approx\mathbf{v}$ holds in the variety $\mathbf{J}_k^r$ if and only if* (2.1)*,* (3.6)*,* (6.34) *and* (6.36) *with $\ell=k$ and $m=r$ hold.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u}\approx\mathbf{v}$ holds in $\mathbf{J}_k^r$. Claims (2.1), (3.6) and (6.34) with $\ell=k$ follow from Proposition 6.14 and the inclusion $\mathbf{I}_k\subseteq\mathbf{J}_k^r$. It remains to verify that (6.36) with $\ell=k$ and $m=r$ holds. As in the proof of Proposition 6.9(i), we can assume that $\mathbf{u}=\mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v}=\mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a}\approx\mathbf{b}\in\{\Phi,\delta_k^r\}$.

If $\mathbf{a}\approx\mathbf{b}\in\Phi$ then (3.6) holds for any ℓ by Proposition 6.9(ii). Evidently, this implies the required conclusion. Suppose now that $\mathbf{a}\approx\mathbf{b}$ coincides with δ_k^r. Then

$$\xi(\mathbf{a})=\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{b}_{r+1}\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$
$$\xi(\mathbf{b})=\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{a}_k\mathbf{b}_{r+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0,\mathbf{a}_1,\dots,\mathbf{a}_k$ and $\mathbf{b}_r,\mathbf{b}_{r+1}$, whence

$$\mathbf{u}=\mathbf{p}\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{b}_{r+1}\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$
$$\mathbf{v}=\mathbf{p}\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{a}_k\mathbf{b}_{r+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a},x_k)=k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_{r+1}$ and $\mathbf{a}_{k-1}$ does not contain any $(k-1)$-divider. Also, obviously, the subword $\mathbf{b}_{r+1}$ of $\mathbf{u}$ located between $\mathbf{b}_r$ and $\mathbf{a}_k$ does not contain any s-divider for all s. Therefore, the subword $\mathbf{b}_{r+1}\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_r$ and $\mathbf{a}_{k-1}$ lies in some $(k-1)$-block. Now Lemma 3.15 again shows that the subword $\mathbf{b}_{r+1}$ located between $\mathbf{p}$ and $\mathbf{b}_r$ does not contain any s-divider for all $s\le r$. Hence if the second occurrence in $\mathbf{u}$ of some letter lies in the subword $\mathbf{b}_{r+1}$ located between $\mathbf{b}_r$ and $\mathbf{a}_k$ then the depth of this letter is more than r. This implies (6.36) with $\ell=k$ and $m=r$.

Sufficiency. As in the proofs of Propositions 6.12 and 6.14, the outline of the argument here is similar to one in the proof of sufficiency in Proposition 6.9(i), but the canonical form of the block is even more complicated than in Proposition 6.14.

Suppose that (2.1), (3.6), (6.34) and (6.36) with $\ell=k$ and $m=r$ all hold. As in the proof of sufficiency in Proposition 6.12, we suppose that (3.4) is the $(k-1)$-decomposition of $\mathbf{u}$, while (6.32) is the representation of $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \ldots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$.

Clearly, $\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. For $j = 0, 1, \ldots, n$, we put

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid \text{the first occurrence of } x \text{ in } \mathbf{u} \text{ lies in } \mathbf{a}_j\}.$$

Let $X_j = \{x_{j1}, x_{j2}, \ldots, x_{jq_j}\}$, $X = X_0 \cup X_1 \cup \cdots \cup X_n$, $\mathbf{a}_j' = (\mathbf{a}_j)_X$. As in the proof of sufficiency in Proposition 6.14, we can verify that

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid s_j = h_1^k(\mathbf{u}, x)\}.$$

For any $j = 0, 1, \ldots, n$, we put

$$Z_j = \{z \in \mathrm{con}(\mathbf{a}_j') \mid D(\mathbf{u}, z) \le r\},$$

$Z = Z_0 \cup Z_1 \cup \cdots \cup Z_n$, $\mathbf{a}_j'' = (\mathbf{a}_j')_Z$ and $\mathbf{u}_i^* = \mathbf{a}_0''\mathbf{a}_1'' \cdots \mathbf{a}_n''$. Let $Z_j = \{z_{j1}, \ldots, z_{jh_j}\}$, $\mathrm{con}(\mathbf{u}_i^*) = \{c_1, \ldots, c_p\}$ and

$$\begin{aligned}\overline{\mathbf{u}_i} = (c_1 \cdots c_p) \cdot (x_{01}^2 \cdots x_{0q_0}^2 z_{01} \cdots z_{0h_0}) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2 z_{11} \cdots z_{1h_1}) \cdots \\ \cdot (s_n x_{n1}^2 \cdots x_{nq_n}^2 z_{n1} \cdots z_{nh_n}).\end{aligned}$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the above mentioned "canonical form" of the $(k-1)$-block $\mathbf{u}_i$.

As in the proof of sufficiency in Proposition 6.14, we can verify that $\mathbf{J}_k^r$ satisfies the identity (6.35). The definitions of the set X and of words of the form $\mathbf{a}_j'$ imply that $z \in \mathrm{con}(\mathbf{w}_1)$ for any $z \in \mathrm{con}(\mathbf{a}_0'\mathbf{a}_1' \cdots \mathbf{a}_n')$. This implies that if $z \in Z_j$ then we can assume that $\mathrm{occ}_z(\mathbf{u}_i) = 1$ because $\mathbf{J}_k^r$ satisfies the identity (6.3) by Lemma 6.2(ii). Then we can assume without loss of generality that $\ell_1(\mathbf{u}, z_{j1}) < \ell_1(\mathbf{u}, z_{j2}) < \cdots < \ell_1(\mathbf{u}, z_{jh_j})$. Since $z \in \mathrm{con}(\mathbf{w}_1)$ and $D(\mathbf{u}, z) > r$ for any $z \in \mathrm{con}(\mathbf{a}_j'')$, we apply Lemma 6.6(i) with $m = r$ to deduce that the identity

$$\begin{aligned}\mathbf{u} \approx \mathbf{w}_1 \cdot \mathbf{u}_i^* \cdot (x_{01}^2 \cdots x_{0q_0}^2 z_{01} \cdots z_{0h_0}) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2 z_{11} \cdots z_{1h_1}) \cdots \\ \cdot (s_n x_{n1}^2 \cdots x_{nq_n}^2 z_{n1} \cdots z_{nh_n}) \cdot \mathbf{w}_2\end{aligned}$$

holds in $\mathbf{J}_k^r$. As we have seen above, $\mathrm{con}(\mathbf{u}_i^*) \subseteq \mathrm{con}(\mathbf{w}_1)$. Then we can apply the identity (6.3) and infer that the word $\mathbf{u}_i^*$ is linear. Then Lemma 6.2(i) shows that $\mathbf{J}_k^r$ satisfies the identities

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}_1 \cdot (c_1 \cdots c_p) \cdot (x_{01}^2 \cdots x_{0q_0}^2 z_{01} \cdots z_{0h_0}) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2 z_{11} \cdots z_{1h_1}) \cdots \\ &\quad \cdot (s_n x_{n1}^2 \cdots x_{nq_n}^2 z_{n1} \cdots z_{nh_n}) \cdot \mathbf{w}_2 \\ &= \mathbf{w}_1 \overline{\mathbf{u}_i}\, \mathbf{w}_2.\end{aligned}$$

So, as in the proof of Proposition 6.9(i), using identities which hold in $\mathbf{J}_k^r$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one by one, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i = m, m-1, \ldots, 0$. Thus $\mathbf{J}_k^r$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp = t_0\, \overline{\mathbf{u}_0}\, t_1\, \overline{\mathbf{u}_1} \cdots t_m\, \overline{\mathbf{u}_m}$.

We turn to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Furthermore, (6.34) with $\ell = k$ and Lemma 3.8 imply that (6.33) is a representation of $t_i\mathbf{v}_i$ as the product of alternating k-dividers $s_0, s_1, \ldots, s_n$ and k-blocks $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$. Claim (6.34) with $\ell = k$ implies that

$$X_j = \{x \in \mathrm{con}(\mathbf{b}_j) \mid s_j = h_1^k(\mathbf{v}, x)\}$$

for all $j = 0, 1, \dots, n$. Put $\mathbf{b}'_j = (\mathbf{b}_j)_X$. In view of (6.36) with $\ell = k$ and $m = r$, we have

$$Z_j = \{z \in \mathrm{con}(\mathbf{b}'_j) \mid D(\mathbf{v}, z) \le r\}$$

for all $j = 0, 1, \dots, n$. Put $\mathbf{b}''_j = (\mathbf{b}'_j)_Z$. Claim (3.6) with $\ell = k$ implies that the jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if the jth occurrence of x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$ for any x and any $j = 1, 2$. Also, Lemma 6.2(ii) allows us to assume that if the first and second occurrences of x in $\mathbf{u}$ do not lie in the $(k-1)$-block $\mathbf{u}_i$ then this letter does not occur in $\mathbf{u}_i$. Thus $\mathrm{con}(\mathbf{u}_i^*) = \mathrm{con}(\mathbf{b}''_0\mathbf{b}''_1 \cdots \mathbf{b}''_n)$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating the arguments above, we find that $\mathbf{J}_k^r$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$. ∎

Now we can complete the proof of Lemma 6.15. Let $\mathbf{I}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$. We have to verify that $\mathbf{X} \supseteq \mathbf{J}_k^1$. Suppose that $\mathbf{J}_k^1 \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{J}_k^1$. Then Propositions 6.14 and 6.17 and the inclusion $\mathbf{I}_k \subset \mathbf{X}$ imply that (2.1), (3.6) and (6.34) hold, while (6.36) with $m = 1$ is false. Then Lemma 6.16(i) implies that $\mathbf{X} \subseteq \mathbf{I}_k$, a contradiction. Lemma 6.15 is proved. ∎

6.4.4. If $\mathbf{J}_k^m \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ with $1 \le m < k$ then $\mathbf{J}_k^{m+1} \subseteq \mathbf{X}$. The fourth step in the verification of Proposition 6.1(4) is

LEMMA 6.18. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{J}_k^m, \mathbf{F}_{k+1}]$ for some $1 \le m < k$ then either $\mathbf{X} = \mathbf{J}_k^m$ or $\mathbf{X} \supseteq \mathbf{J}_k^{m+1}$.*

Proof. Let $1 \le m < k$, $\mathbf{J}_k^m \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ and $\mathbf{J}_k^{m+1} \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{J}_k^{m+1}$. Then Proposition 6.17 and the inclusion $\mathbf{J}_k^m \subset \mathbf{X}$ imply that (2.1), (3.6), (6.34), and (6.36) with $\ell = k$ all hold, while the claim

$$\text{if } x \in \mathrm{con}(\mathbf{u}) \text{ and } D(\mathbf{u}, x) \le m+1 \text{ then } h_2^k(\mathbf{u}, x) = h_2^k(\mathbf{v}, x)$$

is false. Then Lemma 6.16(ii) implies $\mathbf{X} \subseteq \mathbf{J}_k^m$, a contradiction. We see that either $\mathbf{X} = \mathbf{J}_k^m$ or $\mathbf{J}_k^{m+1} \subseteq \mathbf{X}$. ∎

6.4.5. The interval $[\mathbf{J}_k^k, \mathbf{F}_{k+1}]$ consists of $\mathbf{J}_k^k$ and $\mathbf{F}_{k+1}$ only. The fifth step in the verification of Proposition 6.1(4) is

LEMMA 6.19. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{J}_k^k, \mathbf{F}_{k+1}]$ then either $\mathbf{X} = \mathbf{J}_k^k$ or $\mathbf{X} = \mathbf{F}_{k+1}$.*

Proof. Suppose that $\mathbf{J}_k^k \subset \mathbf{X} \subset \mathbf{F}_{k+1}$. Since $\mathbf{F}_{k+1} \nsubseteq \mathbf{X}$, there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but not in $\mathbf{F}_{k+1}$. Propositions 6.9(i) and 6.17 and the inclusion $\mathbf{J}_k^k \subset \mathbf{X}$ imply that (2.1), (3.6), (6.34), and (6.36) with $\ell = m = k$ all hold, while $h_2^k(\mathbf{u}, x) \ne h_2^k(\mathbf{v}, x)$ for some letter $x \in \mathrm{con}(\mathbf{u})$ such that $D(\mathbf{u}, x) > k$. Then we apply Lemma 6.8 for the variety $\mathbf{F}_{k+1}$ and obtain $\mathbf{X} \subseteq \mathbf{J}_k^k$, a contradiction. ∎

6.4.6. All inclusions are strict. Here we are going to verify the inclusions (6.1). To do this, we use Lemma 6.5 and Table 6.1 without explicit mention. We note that the non-strict inclusions (6.5) are true by Lemma 6.4. If $\mathbf{u} \approx \mathbf{v}$ is the identity α_k then $D(\mathbf{u}, x_k) = k$ but $h_1^k(\mathbf{u}, x_k) = \lambda$ and $h_1^k(\mathbf{v}, x_k) = y_k$. Then Proposition 6.12 implies that $\mathbf{F}_k \subset \mathbf{H}_k$. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ coincides with β_k. Then $h_1^k(\mathbf{u}, x) = \lambda$, while $h_1^k(\mathbf{v}, x) = x_k$. We apply Proposition 6.14 to obtain $\mathbf{H}_k \subset \mathbf{I}_k$. Let now $\mathbf{u} \approx \mathbf{v}$

be equal to γ_k. In this case $D(\mathbf{u}, y_1) = 1$ but $h_2^k(\mathbf{u}, y_1) = y_0$ and $h_2^k(\mathbf{v}, y_1) = x_k$. In view of Proposition 6.17, $\mathbf{I}_k \subset \mathbf{J}_k^1$. Suppose now that $\mathbf{u} \approx \mathbf{v}$ coincides with δ_k^m for some $1 \le m < k$. Then $D(\mathbf{u}, y_{m+1}) = m+1$ but $h_2^k(\mathbf{u}, y_{m+1}) = y_m$ and $h_2^k(\mathbf{v}, y_{m+1}) = x_k$. Now we apply Proposition 6.17 again to obtain $\mathbf{J}_k^m \subset \mathbf{J}_k^{m+1}$. Finally, suppose that $\mathbf{u} \approx \mathbf{v}$ is the identity δ_k^k. Since $h_2^k(\mathbf{u}, y_{k+1}) = y_k$ and $h_2^k(\mathbf{v}, y_{k+1}) = x_k$, Proposition 6.9(i) implies that $\mathbf{J}_k^k \subset \mathbf{F}_{k+1}$.

Thus, we have proved the inclusions (6.1). Therefore, the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^1$, $\mathbf{J}_k^2$, …, $\mathbf{J}_k^k$ and $\mathbf{F}_{k+1}$ are pairwise different. This fact and Lemmas 6.10, 6.13, 6.15, 6.18 (with $m = 1, \dots, k-1$) and 6.19 imply Proposition 6.1(4). In view of Lemma 2.10(i) and the results of Sections 6.1 and 6.3, we have completed the proof of Proposition 6.1. ■

Lemmas 2.8 and 2.9(i), Corollary 4.7, Propositions 5.1, 5.2 and 6.1, and the dual of Propositions 5.2 and 6.1 imply the "if" part of Theorem 1.1.

Recall that the "only if" part of Theorem 1.1 was verified in Chapter 4. Thus, Theorem 1.1 is completely proved. ■

7. Corollaries

First of all, we present an exhaustive list of non-group chain varieties of monoids. Theorem 1.1 together with Lemmas 2.8 and 2.9(i), Corollary 4.7, Propositions 5.1, 5.2 and 6.1, and the duals of Propositions 5.2 and 6.1 implies

Corollary 7.1. *The varieties* $\mathbf{C}_n$, $\mathbf{D}_k$, $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{F}_k$, $\overleftarrow{\mathbf{F}_k}$, $\mathbf{H}_k$, $\overleftarrow{\mathbf{H}_k}$, $\mathbf{I}_k$, $\overleftarrow{\mathbf{I}_k}$, $\mathbf{J}_k^m$, $\overleftarrow{\mathbf{J}_k^m}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{M}$, $\overleftarrow{\mathbf{M}}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$, $\mathbf{RRB}$, $\mathbf{SL}$ *where* $n \ge 2$, $k \in \mathbb{N}$ *and* $1 \le m \le k$, *and only these varietes, are non-group chain varieties of monoids.* ■

The set of all non-group chain varieties of monoids ordered by inclusion together with the variety $\mathbf{T}$ is shown in Fig. 7.1. It is interesting to compare this figure with the diagram of the partially ordered set of all non-group chain varieties of semigroups (as already mentioned in Chapter 1, such varieties were completely determined in [22]). This diagram is shown in Fig. 7.2 where $\mathbf{LZ} = \mathrm{var}\{xy \approx x\}$, $\mathbf{RZ} = \mathrm{var}\{xy \approx y\}$, $\mathbf{ZM} = \mathrm{var}\{xy \approx 0\}$, $\mathbf{N}_k = \mathrm{var}\{x^2 \approx x_1 \cdots x_k \approx 0,\ xy \approx yx\}$ for all $k \ge 3$, $\mathbf{N}_\omega = \mathrm{var}\{x^2 \approx 0,\ xy \approx yx\}$, $\mathbf{N}_3^2 = \mathrm{var}\{x^2 \approx xyz \approx 0\}$ and $\mathbf{N}_3^c = \mathrm{var}\{xyz \approx 0,\ xy \approx yx\}$ (here $\mathrm{var}\,\Sigma$ means the semigroup variety given by Σ; as is usual when considering semigroup varieties, we write $\mathbf{w} \approx 0$ as a shorthand for the identity system $\mathbf{w}x \approx x\mathbf{w} \approx \mathbf{w}$ where $x \notin \mathrm{con}(\mathbf{w})$).

We see that, apart from the group case, there is one countably infinite series and six "sporadic" chain semigroup varieties, but ten countably infinite series and twelve "sporadic" chain monoid varieties. Namely, we have the countably infinite series $\mathbf{N}_k$ (including $\mathbf{ZM}$ as $\mathbf{N}_2$) and sporadic varieties $\mathbf{LZ}$, $\mathbf{RZ}$, $\mathbf{SL}$, $\mathbf{N}_3^2$, $\mathbf{N}_3^c$, $\mathbf{N}_\omega$ in the semigroup case, and countably infinite series $\mathbf{C}_n$ (including $\mathbf{SL}$ as $\mathbf{C}_1$), $\mathbf{D}_k$, $\mathbf{F}_k$, $\overleftarrow{\mathbf{F}_k}$, $\mathbf{H}_k$, $\overleftarrow{\mathbf{H}_k}$, $\mathbf{I}_k$, $\overleftarrow{\mathbf{I}_k}$, $\mathbf{J}_k^m$, $\overleftarrow{\mathbf{J}_k^m}$ and sporadic varieties $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{M}$, $\overleftarrow{\mathbf{M}}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$, $\mathbf{RRB}$ in the monoid case. One can say that the number of non-group chain varieties in the case of monoids is much larger (in an informal sense) than in the case of semigroups. Consequently, the partially ordered set of non-group chain varieties in the former case is much more complicated than in the latter.

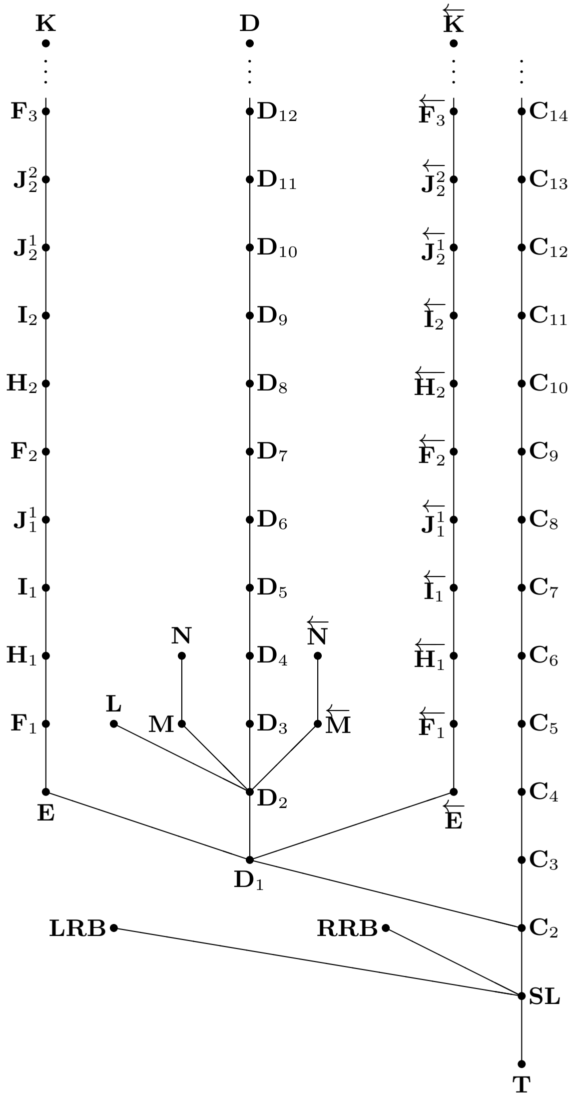

Fig. 7.1. All non-group chain varieties of monoids

As mentioned in Chapter 1, a non-group chain variety of semigroups is contained in a maximal chain variety, while this is not the case for monoid varieties. The following two corollaries indicate cases when the analog of the semigroup statement is true. Fig. 7.1 shows that the following is true.

Corollary 7.2. *A non-group chain variety* $\mathbf{V}$ *of monoids is contained in some maximal chain variety if and only if* $\mathbf{C}_3 \nsubseteq \mathbf{V}$. ∎

Theorem 1.1 shows that commutative non-group chain varieties of monoids are exclusively $\mathbf{SL}$ and $\mathbf{C}_n$ with $n \geq 2$. This claim and Fig. 7.1 imply

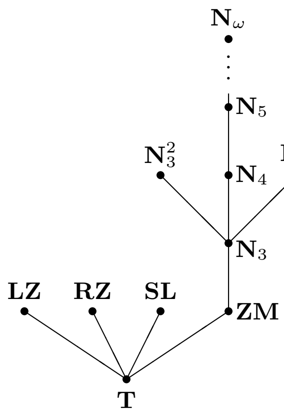

Fig. 7.2. All non-group chain varieties of semigroups

Corollary 7.3. *A non-commutative non-group chain variety of monoids is contained in some maximal chain variety.* ∎

In the following corollary we mention the variety **O** introduced in Section 4.3.

Corollary 7.4. *Let* **X** *be a monoid variety with* $\mathbf{L} \subset \mathbf{X} \subseteq \mathbf{O}$. *Then* **X** *is not a chain variety and does not contain a just-non-chain subvariety.*

Proof. Theorem 1.1 immediately implies that there are no chain monoid varieties that properly contain **L**, whence **X** is not a chain variety. It remains to check that **X** does not contain a just-non-chain subvariety. Suppose that **X** contains such a subvariety **Y**. In view of Theorem 1.1, any chain subvariety of **O** is contained in **L**. In particular, **O** (and therefore **Y**) does not contain incomparable chain subvarieties. On the other hand, being a non-chain variety, **Y** contains at least two incomparable subvarieties. These are proper subvarieties of **Y**, whence they are chain varieties. We have a contradiction. ∎

The following question seems to be interesting.

Question 7.5. Is it true that a non-chain non-group monoid variety **X** with $\mathbf{X} \nsubseteq \mathbf{O}$ contains a just-non-chain subvariety?

Recall that a variety of universal algebras is called *locally finite* if its finitely generated members are all finite. A variety is called *finitely generated* if it is generated by a finite algebra. Clearly, if a variety is contained in some finitely generated variety then it is locally finite.

Corollary 7.6. *An arbitrary non-group chain monoid variety is contained in some finitely generated variety; in particular, it is locally finite.*

Proof. Clearly, it suffices to verify that each of the varieties listed in Theorem 1.1 is contained in a finitely generated variety. It is well known that a proper variety of band monoids is finitely generated [5]. In particular, **LRB** and **RRB** have this property. It is evident that the monoid $S(\mathbf{w})$ is finite for any word **w**. Then Lemmas 2.4 and 4.6 provide the required conclusion for $\mathbf{C}_n$ and $\mathbf{L}$ respectively. The fact that $\overleftarrow{\mathbf{N}}$ is finitely generated follows from [8, Example 1 in Erratum]. By symmetry, it remains to consider the varieties $\mathbf{D}$ and $\mathbf{K}$.

The variety $\mathbf{D}$ is not finitely generated by [14, Theorem 2], but it is shown in [15, Example 5.3] that $\mathbf{D}$ is a subvariety of the variety generated by the well-known 6-element Brandt monoid $B_2^1 = B_2 \cup \{1\}$ where

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\rangle = \{a, b, ab, ba, 0\}.$$

Finally, it is easy to see that if a monoid M belongs to $\mathbf{K}$ and consists of k elements then M satisfies the identity α_k. Therefore, any finitely generated subvariety of $\mathbf{K}$ is contained in $\mathbf{F}_k$ for some k. In particular, $\mathbf{K}$ is not finitely generated. But Lemma 6.2 implies that $\mathbf{K} \subseteq \mathrm{var}\{xyxzx \approx xyxz,\ \sigma_2\}$. To complete our considerations, it remains to note that $\mathrm{var}\{xyxzx \approx xyxz,\ \sigma_2\}$ is generated by the 5-element monoid

$$\langle a, b \mid a^2 = ab = a,\ b^2a = b^2\rangle \cup \{1\} = \{a, b, ba, b^2, 1\}.$$

This is proved in [17, Corollary 6.6]. ∎

The analog of Corollary 7.6 for arbitrary chain varieties of monoids (including group ones) does not hold. Indeed, as mentioned in Chapter 1, it is verified in [11] that there are uncountably many non-locally finite chain varieties of groups. But explicit examples of such varieties have not been specified yet.

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