CONTRIBUTIONS TO GENERAL ALGEBRA 12
Proceedings of the Vienna Conference, June 3–6, 1999
Verlag Johannes Heyn, Klagenfurt 2000

COMMUTING FULLY INVARIANT CONGRUENCES ON FREE SEMIGROUPS

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Abstract. We classify all semigroup varieties on whose free objects the fully invariant congruences contained in the least semilattice congruence commute. It turns out that this property is closely related to the distributive law in subvariety lattices.

Introduction and summary

On every semigroup S, there exists the least congruence σ such that the quotient S/σ is a semilattice (a commutative idempotent semigroup). The congruence σ is called the *least semilattice congruence*, and any congruence contained in σ is said to be a *sub-semilattice* congruence.

A major breakthrough in the theory of semigroup varieties was achieved in the early 90s when Pastijn [8] and Petrich–Reilly [9] independently proved that the sub-semilattice fully invariant congruences on the free completely regular semigroups form a commutative semigroup with respect to the usual relational product. This striking result implies for example that the lattice of completely regular semigroup varieties satisfies the arguesian law. The goal of the present paper is to give a complete description of all semigroup varieties $\mathcal{V}$ such that the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups form a commutative semigroup.

In order to formulate our description, we need some notation. We adopt the usual agreement of writing $w = 0$ as a short form of the identity system $wu = uw = w$ where u runs over the set of all words. By $\operatorname{var}\Sigma$ we denote the variety of all semigroups satisfying the identity system Σ, and $\mathcal{X} \vee \mathcal{Y}$ stands for the *lattice join* of the varieties $\mathcal{X}$ and $\mathcal{Y}$, that is, the least variety

The work was supported by the Russian Foundation of Basic Research (Grant 97-01-00090) and by the program "Russian Universities — Basic Research" of the Ministry of General and Professional Education of the Russian Federation (project No. 617).

containing both $\mathcal{X}$ and $\mathcal{Y}$. Put

$$\mathcal{C} = \mathrm{var}\{x^2 = x^3,\ xy = yx\},$$
$$\mathcal{P} = \mathrm{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\},$$
$$\mathcal{P}^* = \mathrm{var}\{xy = xy^2,\ x^2y^2 = y^2x^2\},$$
$$\mathcal{SL} = \mathrm{var}\{x^2 = x,\ xy = yx\}.$$

Main Theorem. *Let $\mathcal{V}$ be a semigroup variety. The sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups form a commutative semigroup with respect to the relational product if and only if $\mathcal{V}$ satisfies one of the following conditions:*

1) *$\mathcal{V}$ consists of completely regular semigroups, or, equivalently, satisfies the identity $x = x^{n+1}$ for some positive integer n;*
2) *$\mathcal{V} = \mathcal{P}$ or $\mathcal{V} = \mathcal{P}^*$;*
3) *$\mathcal{V}$ contains the variety $\mathcal{SL}$ and satisfies the identity*

$$(xy)^2 = xy; \tag{0.1}$$

4) *$\mathcal{V}$ satisfies one of the following identity systems:*

$$xyz = 0; \tag{0.2}$$
$$xyz = yxz,\ x^2 = 0; \tag{0.3}$$
$$xyz = xzy,\ x^2 = 0; \tag{0.4}$$
$$xyz = yzx,\ x^2 = 0; \tag{0.5}$$
$$xyz = zyx,\ x^2 = xyx = 0; \tag{0.6}$$
$$xyz = zyx,\ x^2 = xyzt = 0; \tag{0.7}$$

5) *$\mathcal{V} = \mathcal{SL} \vee \mathcal{M}$ where $\mathcal{M}$ satisfies one of the following identity systems:*

$$xyz = yxz,\ x^2y = 0; \tag{0.8}$$
$$xyz = yxz,\ xy^2 = 0; \tag{0.9}$$
$$xyz = yxz,\ x^2y = xy^2,\ x^2yz = 0; \tag{0.10}$$
$$xyz = yxz,\ x^2y = yx^2,\ x^3y = 0; \tag{0.11}$$
$$xyz = yxz,\ x^2y = yx^2,\ x^2y^2 = 0; \tag{0.12}$$
$$xyz = yxz,\ x^2y = yx^2,\ x^3y = x^2y^2,\ x^2y^2z = 0; \tag{0.13}$$
$$xyz = xzy,\ x^2y = 0; \tag{0.14}$$
$$xyz = xzy,\ xy^2 = 0; \tag{0.15}$$
$$xyz = xzy,\ x^2y = xy^2,\ x^2yz = 0; \tag{0.16}$$
$$xyz = xzy,\ x^2y = yx^2,\ x^3y = 0; \tag{0.17}$$

$$xyz = xzy,\ x^2y = yx^2,\ x^2y^2 = 0; \tag{0.18}$$

$$xyz = xzy,\ x^2y = yx^2,\ x^3y = x^2y^2,\ x^2y^2z = 0; \tag{0.19}$$

$$xyz = zyx,\ x^2y = 0; \tag{0.20}$$

$$xyz = zyx,\ xyx = 0; \tag{0.21}$$

$$xyz = zyx,\ x^2y = yxy,\ x^2yz = 0; \tag{0.22}$$

$$xyz = zyx,\ x^2y = xyx,\ x^3y = 0; \tag{0.23}$$

$$xyz = zyx,\ x^2y = xyx,\ x^2y^2 = 0; \tag{0.24}$$

$$xyz = zyx,\ x^2y = xyx,\ x^3y = x^2y^2,\ x^2y^2z = 0; \tag{0.25}$$

$$xyz = yzx,\ x^3y = 0; \tag{0.26}$$

$$xyz = yzx,\ x^2y^2 = 0; \tag{0.27}$$

$$xyz = yzx,\ x^3y = x^2y^2,\ x^2y^2z = 0; \tag{0.28}$$

6) *$\mathcal{V} = \mathcal{C} \vee \mathcal{N}$ where $\mathcal{N}$ satisfies the identities*

$$x^2y = xyx = yx^2 = 0 \tag{0.29}$$

and an identity of the kind

$$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi} \tag{0.30}$$

for some non-trivial permutation π.

Following the reasoning from [8] or [9], it is easy to deduce that the subvariety lattice of every variety satisfying the conditions of our theorem is arguesian. Our description, however, reveals the quite surprising fact that the permutability of sub-semilattice fully invariant congruences on free semigroups is closely related to a much stronger lattice identity: namely, the subvariety lattice of every variety satisfying any of the conditions 2)–6) is distributive. Thus, each variety on whose free semigroups the sub-semilattice fully invariant congruences form a commutative semigroup either consists of completely regular semigroups or has a distributive subvariety lattice.

The paper is structured as follows. It has 4 sections. Section 1 contains all necessary preliminaries. In Sections 2 and 3 the "only if" and respectively the "if" parts of the main result are proved. Section 4 is devoted to the just mentioned relationship between the permutability of sub-semilattice fully invariant congruences on free semigroups and the distributive law in the lattices of semigroup varieties.

Finally, to prevent possible confusion, we clarify the connection between the present article and our paper [15]. In that paper, we have described varieties $\mathcal{V}$ such that the sub-semilattice fully invariant congruences on each *relatively free* semigroups in $\mathcal{V}$ commute. The difference between the latter condition and what we are dealing with now is due to the fact that being a relatively free semigroup and a member of $\mathcal{V}$ is not the same as being a $\mathcal{V}$-free semigroup: a relatively free member of $\mathcal{V}$ is a $\mathcal{W}$-free semigroup for some subvariety $\mathcal{W}$ of $\mathcal{V}$. The two conditions are not equivalent (see Example 2.10 in [15]); in fact, as a comparison between the main theorem of the present paper and Theorem 2 in [15] shows, the present condition is much weaker, thus distinguishing a much larger class of semigroup varieties.

1. Preliminaries

1.1. **Some notational conventions.** When speaking about a lattice L, we denote by $\vee$ and $\wedge$ respectively the join and the meet in L, while, in the case when elements of L are sets, $\cup$ means the set-theoretical union. If $x, y \in L$ with $x \leq y$, then $[x, y]$ stands for the interval of L with the bottom element x and the top element y. The subvariety lattice of a variety $\mathcal{V}$ is denoted by $L(\mathcal{V})$, and by $\mathrm{Sub}(G)$ we denote the subgroup lattice of a group G.

Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a fixed infinite sequence of symbols (called *letters*). For any positive integer n, we put $X_n = \{x_1, x_2, \ldots, x_n\}$. Let F be the absolutely free semigroup over X. As usual, elements of F are called *words.* If u is a word then $\ell(u)$ denotes the length of u, $\ell_x(u)$ is the number of occurrences of the letter x in u, $c(u)$ is the set of all letters occurring in u, $n(u) = |c(u)|$, and $t(u)$ is the last letter of u. We call a word u *linear* if $\ell_x(u) = 1$ for every letter $x \in c(u)$. By $\equiv$ we denote the equality relation on F. For $u, v \in F$, we write $u \lhd v$ if $v \equiv a\xi(u)b$ for some endomorphism ξ of F and some $a, b \in F^1$ where F^1 is F with the empty word 1 adjoined. We say that $u, v \in F$ are *similar* and write $u \approx v$ if v can be obtained from u by renaming of letters. Given $u \in F$, the *left indicator* of u is the shortest word v such that $u \equiv vw$ for some $w \in F^1$ and $c(v) = c(u)$. Let v be the left indicator of a word u. Then we denote the letter $t(v)$ by $\tau(u)$ and the word $w \in F^1$ such that $v \equiv w\tau(u)$ by $s(u)$. It is clear that $c(s(u)) = c(u) \setminus \{\tau(u)\}$.

For a semigroup S, an element $x \in S$ and a congruence α on S, we denote by x^α the α-class of S containing x. By $\mathbf{R}$, $\mathbf{L}$ and $\mathbf{H}$ we denote the Green relations on S (cf. [1, §2.1]).

1.2. **Permutation groups and permutation identities.** By $\mathbb{S}_n$ we denote the group of all permutations of the set $\{1, 2, \ldots, n\}$. The subgroup of $\mathbb{S}_n$ generated by $\pi \in \mathbb{S}_n$ is denoted by $\mathrm{gr}\{\pi\}$. For $i \in \{1, 2, \ldots, n\}$, we put

$$\mathrm{Stab}_n(i) = \{\pi \in \mathbb{S}_n \mid i\pi = i\}.$$

Clearly, $\mathrm{Stab}_n(i)$ is a subgroup of $\mathbb{S}_n$. Moreover, it is well known to be a maximum proper subgroup of $\mathbb{S}_n$.

For a positive integer k with $k < n$, we put

$$G_{n,k} = \{\sigma \in \mathbb{S}_n \mid i\sigma \leq k \text{ and } j\sigma > k \text{ whenever } 1 \leq i \leq k < j \leq n\}.$$

Clearly, $G_{n,k}$ is a subgroup of the group $\mathbb{S}_n$. We will need the following simple lemma which proof we include for completeness' sake.

Lemma 1.1. *The interval* $[G_{n,k}, \mathbb{S}_n]$ *of the lattice* $\mathrm{Sub}(\mathbb{S}_n)$ *has at most* 3 *elements.*

Proof. Consider any $\pi \in \mathbb{S}_n$ is such that

$$i\pi \le k \text{ and } j\pi \le k \text{ for some } i, j \text{ with } 1 \le j \le k < i \le n. \tag{1.1}$$

Put $p = i\pi$ and $q = j\pi$. Then $\pi(pq)\pi^{-1} = (ij)$ and, for any r, s with $1 \le s \le k < r \le m$,

$$(rs) = (js)(ij)(ir)(ij)(js).$$

Since the transpositions $(pq), (js), (ir)$ lie in $G_{n,k}$, we have $(rs) \in G_{n,k} \vee \mathrm{gr}\{\pi\}$. Hence the group $G_{n,k} \vee \mathrm{gr}\{\pi\}$ contains all transpositions, and therefore, $G_{n,k} \vee \mathrm{gr}\{\pi\} = \mathbb{S}_n$. Analogously, $\mathbb{S}_n = G_{n,k} \vee \mathrm{gr}\{\pi\}$ for any permutation $\pi \in \mathbb{S}_n$ such that

$$i\pi > k \text{ and } j\pi > k \text{ for some } i, j \text{ with } 1 \le j \le k < i \le n. \tag{1.2}$$

Put $H = G_{n,k} \cup R$ where

$$R = \{\pi \in \mathbb{S}_n \mid \pi \notin G_{n,k} \text{ and } \pi \text{ satisfies neither (1.1) nor (1.2)}\}.$$

It is clear that if $\phi, \xi \in R$ and $\psi \in G_{n,k}$ then $\phi\xi \in G_{n,k}$ and $\phi\psi, \psi\phi \in R$. In particular, H is a subgroup of $\mathbb{S}_n$. As we proved above, there is no subgroup Q with $H \subset Q \subset \mathbb{S}_n$. Furthermore, it is easy to see that, for any $\phi, \xi \in R$, there exists $\psi \in G_{m,k}$ with $\xi = \phi\psi$. This means that $H = G_{n,k} \vee \mathrm{gr}\{\pi\}$ for any $\pi \in R$. Hence there is no subgroup P with $G_{n,k} \subset P \subset H$. Thus, the interval $[G_{n,k}, \mathbb{S}_n]$ contains, besides its extremes, only the subgroup H (which may coincide with one of the extremes), and hence, has ≤ 3 elements. □

Recall that a semigroup variety is called *permutational* if it satisfies a non-trivial *permutation identity*, that is, an identity of the kind

$$x_1 x_2 \cdots x_n = x_{1\pi} x_{2\pi} \cdots x_{n\pi} \tag{1.3}$$

for some $\pi \in \mathbb{S}_n$. The number n is called the *length* of the identity (1.3). For a semigroup variety $\mathcal{V}$ and a positive integer n, we put

$$\mathrm{Perm}_n(\mathcal{V}) = \{\pi \in \mathbb{S}_n \mid \mathcal{V} \text{ satisfies the identity (1.3)}\}.$$

Clearly, $\mathrm{Perm}_n(\mathcal{V})$ is a subgroup of $\mathbb{S}_n$. Gy. Pollák [11] has studied in depth the relationship between the groups $\mathrm{Perm}_n(\mathcal{V})$ with different n (and fixed $\mathcal{V}$). We will make use of the following partial case of his results:

Lemma 1.2. *Suppose that a semigroup variety* $\mathcal{V}$ *satisfies an identity of the kind* (0.30) *for some non-trivial permutation* π. *Then for* $n \ge 4$

- **a)** $\mathrm{Perm}_n(\mathcal{V}) \supseteq \mathrm{Stab}_n(n)$ *whenever* $\pi = (12)$;
- **b)** $\mathrm{Perm}_n(\mathcal{V}) \supseteq \mathrm{Stab}_n(1)$ *whenever* $\pi = (23)$;
- **c)** $\mathrm{Perm}_n(\mathcal{V}) = \mathbb{S}_n$ *otherwise.* □

1.3. **Nil-varieties and their identities.** Recall that a semigroup S is said to be *nilpotent* if S satisfies the identity $x_1 \cdots x_n = 0$ for some n. A semigroup variety $\mathcal{V}$ is a *locally nilpotent variety* (respectively a *nil-variety*) if each finitely generated (one-generated) member of $\mathcal{V}$ is nilpotent. We will often use the following technical remarks about identities of nil-varieties.

Lemma 1.3. **(i)** *If a nil-variety $\mathcal{V}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$, then $\mathcal{V}$ satisfies also the identity $u = 0$.*
(ii) *If a nil-variety $\mathcal{V}$ satisfies an identity of the form $x_1 x_2 \cdots x_n = v$ with $\ell(v) \neq n$, then it satisfies also the identity $x_1 x_2 \cdots x_n = 0$.*
(iii) *If a locally nilpotent variety $\mathcal{N}$ satisfies an identity $u = v$ with $\ell(u) < \ell(v)$ and $u \lhd v$, then $\mathcal{N}$ satisfies also the identity $u = 0$.*

Proof. (i) We may assume that there is a letter $x \in c(v) \setminus c(u)$. Substituting 0 for x, we obtain $u = 0$.

(ii) If $\ell(v) < n$, then $c(v) \neq X_n$ and the statement (i) applies. If $\ell(v) > n$, then the claim follows from [12, Lemma 1].

(iii) Since $u \lhd v$, there exist $a_0, b_0 \in F^1$ and an endomorphism ξ of F such that $v \equiv a_0 \xi(u) b_0$. For any positive integer k, we recursively put

$$a_k \equiv a_{k-1} \xi^k(a_0), \quad b_k \equiv \xi^k(b_0) b_{k-1} \quad \text{and} \quad u_k \equiv a_{k-1} \xi^k(u) b_{k-1}$$

(of course, we assume that $\xi(1) = 1$). On the free semigroup F, we consider the relation ν such that $w_1 \nu w_2$ if and only if the identity $w_1 = w_2$ holds in $\mathcal{N}$. Clearly, ν is a fully invariant congruence. We then have

$$\begin{aligned} u \ \nu \ v \equiv & \underbrace{a_0 \xi(u) b_0}_{u_1} \ \nu \ a_0 \xi(v) b_0 \equiv a_0 \xi(a_0 \xi(u) b_0) b_0 \equiv \\ & \underbrace{a_1 \xi^2(u) b_1}_{u_2} \ \nu \ a_1 \xi^2(v) b_1 \equiv a_1 \xi^2(a_0 \xi(u) b_0) b_1 \equiv \\ & \underbrace{a_2 \xi^3(u) b_2}_{u_3} \ \nu \ a_2 \xi^3(v) b_2 \equiv a_2 \xi^3(a_0 \xi(u) b_0) b_2 \equiv \cdots \equiv \\ & \underbrace{a_{k-1} \xi^k(u) b_{k-1}}_{u_k} \ \nu \ a_{k-1} \xi^k(v) b_{k-1} \equiv a_{k-1} \xi^k(a_0 \xi(u) b_0) b_{k-1} \equiv \cdots . \end{aligned}$$

We see that $u \nu u_k$ for any k. By (i), we may also assume that $c(u) = c(u_k)$ for all k.

First suppose that $a_0 = b_0 = 1$. Then $v \equiv \xi(u)$. Since $\ell(u) < \ell(v)$, there is a letter $x \in c(u)$ such that $\ell(\xi(x)) > 1$. Furthermore, $u_{k+1} = \xi(u_k)$, whence $\ell(u_k) < \ell(\xi(u_k)) = \ell(u_{k+1})$ for each k. Thus,

$$\ell(u) < \ell(u_1) < \ell(u_2) < \cdots < \ell(u_k) < \cdots . \tag{1.4}$$

Now suppose that $a_0 \neq 1$. Then it is evident that $\ell(a_{k-1}) < \ell(a_k)$ for every k. Since $\ell(b_{k-1}) \leq \ell(b_k)$ and $\ell(u_k) \leq \ell(u_{k+1})$, the inequalities (1.4) hold true again. By symmetry, they also hold whenever $b_0 \neq 1$.

Let $N = F/\nu$ and let $\psi : F \to N$ be an arbitrary homomorphism. Since the variety $\mathcal{N}$ is locally nilpotent, the subsemigroup of N generated by the set $\{\psi(x) \mid x \in c(u)\}$ satisfies $x_1 x_2 \cdots x_n = 0$ for some n. From (1.4) it follows that $\ell(u_n) > n$ whence $\psi(u_n) = 0$ in N. Since ψ is an arbitrary homomorphism, the identity $u_n = 0$ holds in N, but since $u \; \nu \; u_n$, we also have $u = 0$ in N. However N is nothing but the $\mathcal{N}$-free semigroup over the infinite set X, and therefore, any identity of N is an identity of the whole variety $\mathcal{N}$. □

We do not know if Lemma 1.3(iii) remains true for an arbitrary nil-variety. In this paper, it will be applied to permutational nil-varieties only, and such varieties are easily seen to be locally nilpotent.

1.4. **G-sets and the subvariety lattice of a nil-variety.** Let m and n be positive integers with $m \le n$. We say that a nil-variety $\mathcal{V}$ is (n,m)-*split* if every identity $u = v$ with $\ell(u) = n$, $n(u) = m$ and $\ell(v) > n$ implies in $\mathcal{V}$ the identity $u = 0$. Further, we say that $\mathcal{V}$ is n-*split* if it is (n,m)-split for any $m \le n$. A nil-variety $\mathcal{V}$ is called *homogeneous* if every identity $u = v$ with $\ell(u) \ne \ell(v)$ implies in $\mathcal{V}$ the identity $u = 0$. Clearly, a variety is homogeneous if and only if it is n-split for any n. A variety is called *hereditarily homogeneous* if all its subvarieties are homogeneous.

We will employ the structure theory of the subvariety lattice of a nil-variety which we have developed in [16]. The theory is based on the notion of a G-set. Let A be a non-empty set, G a group and φ a homomorphism from G into the group of all permutations of A. For every $g \in G$, we define the unary operation g^* on the set A by letting $g^*(a) = (\varphi(g))(a)$ for every $a \in A$. The unary algebra with the carrier set A and the operations $\{g^* \mid g \in G\}$ is called a *G-set*. As usual, $\mathrm{Con}(A)$ denotes the congruence lattice of A.

Following [16], we assign a countable series of G-sets to a given nil-variety $\mathcal{V}$. Let m and n be positive integers with $m \le n$. Put

$$F_{n,m}(\mathcal{V}) = \{u \in F \mid \ell(u) = n,\ c(u) = X_m \text{ and } u \ne 0 \text{ in } \mathcal{V}\}$$

(here and below we write "$u \ne 0$ in $\mathcal{V}$" to express the fact that the identity $u = 0$ fails in $\mathcal{V}$). Fix a subset $W_{n,m}(\mathcal{V})$ of $F_{n,m}(\mathcal{V})$ such that, for each word $u \in F_{n,m}(\mathcal{V})$, there exists a unique word $u^* \in W_{n,m}(\mathcal{V})$ with $u = u^*$ in $\mathcal{V}$. Observe that $u \equiv u^*$ whenever $u \in W_{n,m}(\mathcal{V})$, and $u^* \equiv v^*$ whenever $u = v$ in $\mathcal{V}$. Put

$$W^0_{n,m}(\mathcal{V}) = W_{n,m}(\mathcal{V}) \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ stands for an arbitrary but fixed word w such that $c(w) = X_m$ and $w = 0$ in $\mathcal{V}$.

If $u \in F$, $c(u) = X_m$ and $\psi \in \mathbb{S}_m$, then we denote by $u\psi$ the image of u under the automorphism of F that extends the mapping $x_i \mapsto x_{i\psi}$; we assume here that $i\psi = i$ whenever $i > m$. It is clear that if $u \in F_{n,m}(\mathcal{V})$ and $\psi \in \mathbb{S}_m$

then $u\psi \in F_{n,m}(\mathcal{V})$, and hence, the word $(u\psi)^*$ is well defined. For every $\psi \in \mathbb{S}_m$, we define the unary operation ψ^* on $W^0_{n,m}(\mathcal{V})$ by letting

$$\psi^*(u) \equiv (u\psi)^* \text{ for any } u \in W_{n,m}(\mathcal{V}) \text{ and } \psi^*(\mathbf{0}) \equiv \mathbf{0}.$$

It is easy to verify that the set $W^0_{n,m}(\mathcal{V})$ with the collection of unary operations $\{\psi^* \mid \psi \in \mathbb{S}_m\}$ is an $\mathbb{S}_m$-set whenever $\mathcal{V}$ is (n,m)-split; in particular, if $\mathcal{V}$ is n-split, then $W^0_{n,m}(\mathcal{V})$ is an $\mathbb{S}_m$-set for any $m \le n$, and if $\mathcal{V}$ is homogeneous, then $W^0_{n,m}(\mathcal{V})$ is an $\mathbb{S}_m$-set for all m and n (see [16, Lemma 1.1]). One can note that $W_{n,m}(\mathcal{V})$ is an $\mathbb{S}_m$-subset of $W^0_{n,m}(\mathcal{V})$ whenever $W_{n,m}(\mathcal{V}) \ne \varnothing$. For brevity, we call the $\mathbb{S}_m$-sets $W_{n,m}(\mathcal{V})$ and $W^0_{n,m}(\mathcal{V})$ respectively *transversals* and 0-*transversals*.

By [16, Theorem 1.3], if a nil-variety $\mathcal{V}$ is (n,m)-split, then the lattice $\mathrm{Con}(W^0_{n,m}(\mathcal{V}))$ is dually isomorphic to a certain interval of the lattice $L(\mathcal{V})$, and if $\mathcal{V}$ is hereditarily homogeneous, then $L(\mathcal{V})$ decomposes into a subdirect product of all those intervals. Thus, we have

Proposition 1.4 ([16, Corollary 1.1]). *The subvariety lattice of an arbitrary hereditarily homogeneous semigroup variety $\mathcal{V}$ is dually isomorphic to a subdirect product of the congruence lattices of all* 0*-transversals $W^0_{n,m}(\mathcal{V})$.* □

In [13], the first author has studied congruence permutable and congruence distributive G-sets. In the present paper, we will apply the results of [13] to the 0-transversals $W^0_{n,m}(\mathcal{V})$. Recall that a G-set A is called *transitive* if, for all $a, b \in A$, there exists an element $g \in G$ with $g^*(a) = b$. A transitive G-subset of a G-set A is called an *orbit* of A.

Lemma 1.5. *Let $\mathcal{M}$ be a (n,m)-split variety for some positive integers m and n with $m \le n$. If the $\mathbb{S}_m$-set $W^0_{n,m}(\mathcal{V})$ is either congruence permutable or congruence distributive and $W_{n,m}(\mathcal{V}) \ne \varnothing$, then the $\mathbb{S}_m$-set $W_{n,m}(\mathcal{V})$ is transitive.*

Proof. According to [13, Corollary 2.5 and Theorem 3.4], every congruence permutable or congruence distributive G-set contains ≤ 2 orbits. It remains to take into account that the singleton set $\mathbf{0}$ is always an orbit of $W^0_{n,m}(\mathcal{V})$. □

We need also a well known characterization of the congruence lattice of a transitive G-set. If A is a G-set and $a \in A$ then we put

$$\mathrm{Stab}_A(a) = \{g \in G \mid g^*(a) = a\}.$$

It is clear that $\mathrm{Stab}_A(a)$ is a subgroup of G.

Lemma 1.6 (see [6, Lemma 4.20]). *If A is a transitive G-set and $a \in A$, then the lattice $\mathrm{Con}(A)$ is isomorphic to the interval $[\mathrm{Stab}_A(a), G]$ of the lattice $\mathrm{Sub}(G)$.* □

Corollary 1.7. *Let $\mathcal{V}$ be a nil-variety and n a positive integer. Then the variety $\mathcal{V}$ is (n,n)-split, and if $W_{n,n}(\mathcal{V}) \neq \varnothing$, then the lattice* $\mathrm{Con}(W_{n,n}(\mathcal{V}))$ *is isomorphic to the interval* $[\mathrm{Perm}_n(\mathcal{V}), \mathbb{S}_n]$ *of the lattice* $\mathrm{Sub}(\mathbb{S}_n)$.

Proof. The fact that $\mathcal{V}$ is (n,n)-split immediately follows from Lemma 1.3(ii). Let us put $W = W_{n,n}(\mathcal{V})$ and suppose that $W \neq \varnothing$. Then W is an $\mathbb{S}_n$-set. Clearly, any word of W is similar to $x_1x_2\cdots x_n$. In particular, this implies that W is transitive. We may assume without any loss that $x_1x_2\cdots x_n \in W$. By Lemma 1.6, it remains to verify that $\mathrm{Stab}_W(x_1x_2\cdots x_n) = \mathrm{Perm}_n(\mathcal{V})$. Indeed, let $\psi \in \mathrm{Stab}_W(x_1x_2\cdots x_n)$. This means that

$$x_1x_2\cdots x_n \equiv \psi^*(x_1x_2\cdots x_n) \equiv (x_{1\psi}x_{2\psi}\cdots x_{n\psi})^*.$$

Since the variety $\mathcal{V}$ satisfies the identity $(x_{1\psi}x_{2\psi}\cdots x_{n\psi})^* = x_{1\psi}x_{2\psi}\cdots x_{n\psi}$, we see that the identity $x_1x_2\cdots x_n = x_{1\psi}x_{2\psi}\cdots x_{n\psi}$ holds in $\mathcal{V}$, whence $\psi \in \mathrm{Perm}_n(\mathcal{V})$. Thus, $\mathrm{Stab}_W(x_1x_2\cdots x_n) \subseteq \mathrm{Perm}_n(\mathcal{V})$. Conversely, let $\psi \in \mathrm{Perm}_n(\mathcal{V})$, that is, let $\mathcal{V}$ satisfy the identity $x_1x_2\cdots x_n = x_{1\psi}x_{2\psi}\cdots x_{n\psi}$. Taking into account that $x_1x_2\cdots x_n \in W$, we obtain

$$x_1x_2\cdots x_n \equiv (x_1x_2\cdots x_n)^* \equiv (x_{1\psi}x_{2\psi}\cdots x_{n\psi})^* \equiv \psi^*(x_1x_2\cdots x_n).$$

Thus, $x_1x_2\cdots x_n \equiv \psi^*(x_1x_2\cdots x_n)$, and therefore, $\psi \in \mathrm{Stab}_W(x_1x_2\cdots x_n)$. Hence $\mathrm{Perm}_n(\mathcal{V}) \subseteq \mathrm{Stab}_W(x_1x_2\cdots x_n)$. □

1.5. **Identities of certain semigroup varieties.** We will need a description of the identities of a few concrete semigroup varieties. The statements (i)–(iv) of the following lemma are well known and can be easily verified. The statement (v) was proved in [3, Lemma 7]. Put $\mathcal{ZM} = \{xy = 0\}$, $\mathcal{RZ} = \{xy = y\}$.

Lemma 1.8. *The identity $u = v$ holds in the variety:*
- **(i)** $\mathcal{SL}$ *if and only if $c(u) = c(v)$;*
- **(ii)** $\mathcal{RZ}$ *if and only if $t(u) = t(v)$;*
- **(iii)** $\mathcal{ZM}$ *if and only if either both u and v coincide with the same letter or $\ell(u) > 1$ and $\ell(v) > 1$;*
- **(iv)** $\mathcal{C}$ *if and only if $c(u) = c(v)$ and, for every letter $x \in c(u)$, either $\ell_x(u) > 1$ and $\ell_x(v) > 1$ or $\ell_x(u) = \ell_x(v) = 1$;*
- **(v)** $\mathcal{P}$ *if and only if $c(u) = c(v)$ and either $\ell_{t(u)}(u) > 1$ and $\ell_{t(v)}(v) > 1$ or $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ and $t(u) \equiv t(v)$.* □

The following result is a part of the semigroup folklore. In fact, it easily follows from Lemma 1.8(iii).

Lemma 1.9. *A variety consists of completely regular semigroups if and only if it does not contain the variety $\mathcal{ZM}$.* □

1.6. **Varieties whose subvariety lattice is modular.** As was mentioned in the introduction, varieties which we are interested in have modular subvariety lattices. This suggests to deduce the "only if" part of the main theorem from a complete description of semigroup varieties with the latter property found by the second author. The description, however, is quite involved, and even its precise formulation (see [19, Theorems 1–3]) would require several pages. The good news is that we do not need the description in its full strength, and the following necessary condition will quite suffice for our purposes. (This necessary condition can be easily extracted from results of [17].) Here and below $\mathcal{T}$ stands for the trivial variety containing only the one-element semigroup.

Proposition 1.10. *If the lattice $L(\mathcal{V})$ is modular, then the variety $\mathcal{V}$ must satisfy one of the following conditions:*

a) *for some integer $n>1$, one of the identity systems*

$$(xy)^n = xy, \tag{1.5}$$

$$x^ny = xy,\ (xy)^n = xy^n,\ xyzt = xyx^{n-1}zt, \tag{1.6}$$

$$xy^n = xy,\ (xy)^n = x^ny,\ xyzt = xyt^{n-1}zt, \tag{1.7}$$

holds in $\mathcal{V}$;

b) *$\mathcal{V} = \mathcal{G}\vee\mathcal{X}\vee\mathcal{M}$ where $\mathcal{X}$ is one of the varieties $\mathcal{C}$, $\mathcal{SL}$ or $\mathcal{T}$, $\mathcal{G}$ consists of periodic abelian groups and $\mathcal{M}$ satisfies the identities* (0.29) *and a non-trivial permutation identity of length* 4*;*

c) *$\mathcal{V} = \mathcal{Y}\vee\mathcal{N}$ where $\mathcal{Y}$ is one of the varieties $\mathcal{SL}$ or $\mathcal{T}$ and $\mathcal{N}$ is a nil-variety and satisfies a permutation identity of length* 4 *for some non-trivial even permutation.* □

1.7. **A lifting lemma.** We conclude the preliminary section with a fairly general remark which can be straightforwardly checked.

Lemma 1.11. *Let α,β and ν be equivalences on a set S such that $\alpha,\beta\supseteq\nu$. Then α and β commute* (*the product of α and β coincides with their set-theoretical union*) *if and only if the equivalences α/ν and β/ν on the quotient set S/ν commute* (*respectively, the product of α/ν and β/ν coincides with their set-theoretical union*). □

Lemma 1.11 shows that, when studying commuting fully invariant congruences, we may consider congruences on the absolutely free semigroup F that contain a given fully invariant congruence ν instead of congruences on the relatively free semigroup F/ν. It is convenient for it is easier to deal with words of F than with elements of an arbitrary relatively free semigroup.

2. Necessity

In this section $\mathcal{V}$ always denotes a semigroup variety such that the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups form a commutative *semigroup with respect to the relational product.* We want to prove that $\mathcal{V}$ satisfies one of the conditions 1)–6) of the main theorem.

Let us start the proof with isolating the easiest case. We mean the case when $\mathcal{V} \not\supseteq \mathcal{SL}$ and hence the least semilattice congruence on any semigroup in $\mathcal{V}$ is the universal relation. Clearly, for such a variety $\mathcal{V}$, the condition that the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups commute means in fact that all fully invariant congruences on those semigroups do. The latter restriction has been studied in [15], and from [15, Theorem 1], we immediately obtain

Proposition 2.1. *If $\mathcal{V} \not\supseteq \mathcal{SL}$, then $\mathcal{V}$ satisfies one of the conditions* 1) *or* 4) *of the main theorem.* □

The next alternative will also prove to be quite useful.

Proposition 2.2. *Either $\mathcal{V}$ consists of completely regular semigroups or all groups in $\mathcal{V}$ are trivial.*

Proof. In view of Lemma 1.9 we may assume that $\mathcal{V}$ contains the variety $\mathcal{ZM}$. Let F_1 be the free semigroup with one free generator x and let ν denote the fully invariant congruence on F_1 corresponding to the variety $\mathcal{V}$. Since every one-generator semilattice consists of one element, the least semilattice congruence on F_1 is the universal relation. Then from Lemma 1.11 it follows that all fully invariant congruences on F_1 containing ν commute.

Take an arbitrary finite group $G \in \mathcal{V}$ and denote by ζ and γ the fully invariant congruences on F_1 corresponding to the variety $\mathcal{ZM}$ and respectively the variety $\mathcal{G}$ generated by G. Then both ζ and γ contain ν, whence $\zeta\gamma = \gamma\zeta$.

The variety $\mathcal{G}$ satisfies the identity $x = x^{n+1}$ for some positive integer n. Since $x^{n+1} = x^2$ in $\mathcal{ZM}$, we have $x \mathrel{\gamma} x^{n+1} \mathrel{\zeta} x^2$, that is, $(x, x^2) \in \gamma\zeta$. Hence $(x, x^2) \in \zeta\gamma$, that is, $x \mathrel{\zeta} u \mathrel{\gamma} x^2$ for some $u \in F_1$. Thus, $\mathcal{ZM}$ satisfies the identity $x = u$. By Lemma 1.8(iii), it means that $u \equiv x$. Hence $x \mathrel{\gamma} x^2$, that is, $x = x^2$ in $\mathcal{G}$. Therefore, the group G is trivial. Clearly, then every group in $\mathcal{V}$ is trivial as well. □

Our next observation was already mentioned twice.

Lemma 2.3. *The lattice $L(\mathcal{V})$ is arguesian* (*and therefore, modular*).

Proof. From a result by Mel'nik [7], it follows that the variety $\mathcal{SL}$ is a neutral element of the lattice of semigroup varieties. Hence, the lattice $L(\mathcal{V})$ embeds into the direct product of the interval $[\mathcal{SL} \wedge \mathcal{V}, \mathcal{V}]$ with the lattice $L(\mathcal{SL})$, the latter being the two-element chain. The former interval, however, is dually isomorphic to the lattice of the sub-semilattice fully invariant congruences on the $\mathcal{V}$-free semigroup over the set X. Since lattices of commuting equivalences are known to be arguesian (cf. [4, §IV.4]), and since the arguesian law has been proved to be self-dual [5], we conclude that the interval $[\mathcal{SL}\wedge\mathcal{V},\mathcal{V}]$ is arguesian, and so is the lattice $L(\mathcal{V})$. $\square$

Lemma 2.3 shows that we may select varieties with the desired property among varieties satisfying one of the conditions a)–c) of Proposition 1.10.

Proposition 2.4. *If $\mathcal{V}$ satisfies the condition* a) *of Proposition* 1.10*, then $\mathcal{V}$ satisfies one of the conditions* 1)–4) *of the main theorem.*

Proof. In view of Proposition 2.2, we may assume that all groups in $\mathcal{V}$ are trivial. We may also assume that $\mathcal{V}\supseteq\mathcal{SL}$: otherwise Proposition 2.1 applies. Now consider three cases.

Case 1: *for some integer $n>1$, $\mathcal{V}$ satisfies the identity* (1.5). Let $S\in\mathcal{V}$ and $s,t\in S$. Since all groups in $\mathcal{V}$ are trivial, any group element of S should be an idempotent. In view of the identity (1.5), st is a group element, whence st is an idempotent. Therefore, the variety $\mathcal{V}$ satisfies the identity (0.1). Thus, we have arrived at the condition 3).

Case 2: *for some integer $n>1$, $\mathcal{V}$ satisfies the identity system* (1.6). Put

$$\mathcal{Q}=\mathrm{var}\{xy=x^2y,\ xyx=yx^2,\ xyz^2=yxz^2\}.$$

By [18, Lemma 14], $\mathcal{V}=\mathcal{A}\vee\mathcal{B}$ where $\mathcal{A}$ is one of the varieties $\mathcal{T}$, $\mathcal{ZM}$, $\mathcal{P}$ or $\mathcal{Q}$, and $\mathcal{B}$ consists of completely regular semigroups. Since all groups in $\mathcal{V}$ are trivial, $\mathcal{B}$ consists of idempotent semigroups. Clearly, we may assume without any loss that $\mathcal{B}=\mathcal{V}\wedge\mathrm{var}\{x^2=x\}$ whence, in particular, $\mathcal{B}\supseteq\mathcal{SL}$. If $\mathcal{A}=\mathcal{T}$ or $\mathcal{A}=\mathcal{ZM}$ then $\mathcal{V}$ obviously satisfies the identity (0.1). Thus, we may also assume that $\mathcal{A}=\mathcal{P}$ or $\mathcal{A}=\mathcal{Q}$.

Let ν,ρ,β denote the fully invariant congruences on F corresponding to the varieties $\mathcal{V}$, $\mathcal{P}$ and $\mathcal{B}$, respectively. Clearly, ρ and β contain ν and are contained in the least semilattice congruence on F. By Lemma 1.11 $\rho\beta=\beta\rho$. Observe that $(x_1^2x_2^2,x_2x_1)\in\rho\beta$ because $x_1^2x_2^2\ \rho\ x_2^2x_1^2\ \beta\ x_2x_1$. Therefore, $(x_1^2x_2^2,x_2x_1)\in\beta\rho$, that is, $x_1^2x_2^2\ \beta\ u\ \rho\ x_2x_1$ for some word $u\in F$. Hence $u=x_2x_1$ in $\mathcal{P}$, and by Lemma 1.8(v), $c(u)=\{x_1,x_2\}$, $\ell_{t(u)}(u)=1$ and $t(u)\equiv x_1$. This means that $u\equiv x_2^kx_1$ for some positive integer k. Thus, $x^2y^2=y^kx$ in $\mathcal{B}$. Since $\mathcal{B}$ satisfies $x^2=x$, we conclude that $xy=yx$ in $\mathcal{B}$. Therefore,

$$\mathcal{B}\subseteq\mathcal{SL}=\mathrm{var}\{x^2=x,\ xy=yx\}. \tag{2.1}$$

Since by Lemma 1.8(i) and (v) $\mathcal{SL}\subseteq\mathcal{P}\subseteq\mathcal{A}$, the inclusion (2.1) implies that $\mathcal{B}\subseteq\mathcal{A}$, whence $\mathcal{V}=\mathcal{A}\vee\mathcal{B}=\mathcal{A}$. Thus, either $\mathcal{V}=\mathcal{P}$ or $\mathcal{V}=\mathcal{Q}$.

In the latter case, Lemma 1.8(ii) shows that $\mathcal{V}$ contains the variety $\mathcal{RZ}=\mathrm{var}\{xy=y\}$. However, since $\mathcal{RZ}$ consists of idempotent semigroups, we must have $\mathcal{RZ}\subseteq\mathcal{B}$, while $\mathcal{RZ}\nsubseteq\mathcal{SL}$ in a contradiction to the inclusion (2.1). Therefore, $\mathcal{V}=\mathcal{P}$, and the condition 2) of the main theorem holds true.

Case 3: *for some integer* $n > 1$*,* $\mathcal{V}$ *satisfies the identity system* (1.7). This case is dual to the previous one. $\square$

A semigroup variety is said to be 3-*commutative* if it satisfies all permutation identities of length 3.

Lemma 2.5. *Let* $\mathcal{V} = \mathcal{K} \vee \mathcal{M}$ *where* $\mathcal{K}$ *is a* 3*-commutative variety with* $\mathcal{K} \supseteq \mathcal{SL}$ *and* $\mathcal{M}$ *is a nil-variety. Then* $\mathcal{M}$ *satisfies the identity* (0.30) *for some non-trivial permutation* π*.*

Proof. Arguing by contradiction, suppose that $\mathcal{M}$ satisfies no non-trivial identity of the form (0.30). Let $\mathcal{L}$ and $\mathcal{R}$ denote the subvarieties of $\mathcal{M}$ defined within $\mathcal{M}$ by the identities $xyz = yxz$ and $xyz = xzy$ respectively. We denote by $\nu, \lambda, \rho, \varkappa$ the fully invariant congruences on F corresponding to $\mathcal{V}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{K}$ respectively. Clearly, λ, ρ and $\varkappa$ contain ν. Observe that $\varkappa$ is contained in the least semilattice congruence on F because $\mathcal{SL} \subseteq \mathcal{K}$ by the hypothesis of the lemma. Put $\lambda' = \lambda \wedge \varkappa$ and $\rho' = \rho \wedge \varkappa$. Then λ' and ρ' contain ν and are contained in the least semilattice congruence on F. By Lemma 1.11 $\lambda'\rho' = \rho'\lambda'$.

The variety $\mathcal{K}$ is 3-commutative. Therefore, $x_1x_2x_3 \mathrel{\varkappa} x_2x_1x_3 \mathrel{\varkappa} x_2x_3x_1$, whence $x_1x_2x_3 \mathrel{\lambda'} x_2x_1x_3 \mathrel{\rho'} x_2x_3x_1$. It means that $(x_1x_2x_3, x_2x_3x_1) \in \lambda'\rho'$, and thus, $(x_1x_2x_3, x_2x_3x_1) \in \rho'\lambda' \subseteq \rho\lambda$. It implies that $x_1x_2x_3 \mathrel{\rho} u \mathrel{\lambda} x_2x_3x_1$ for some word $u \in F$. Obviously, $\mathrm{Perm}_3(\mathcal{R}) = \mathrm{Stab}_3(1)$; in particular, $xyz \neq 0$ in $\mathcal{R}$. We then have $c(u) = \{x_1, x_2, x_3\}$ by Lemma 1.3(i) and $\ell(u) = 3$ by Lemma 1.3(ii). Hence either $u \equiv x_1x_2x_3$ or $u \equiv x_1x_3x_2$. Since the variety $\mathcal{L}$ satisfies none of the identities $x_1x_2x_3 = x_2x_3x_1$ and $x_1x_3x_2 = x_2x_3x_1$, either of the possibilities for u leads to a contradiction. $\square$

Proposition 2.6. *If* $\mathcal{V}$ *satisfies the condition* b) *of Proposition* 1.10*, then* $\mathcal{V}$ *satisfies one of the conditions* 1), 4), 5) *or* 6) *of the main theorem.*

Proof. The condition b) of Proposition 1.10 means that $\mathcal{V} = \mathcal{G} \vee \mathcal{X} \vee \mathcal{M}$ where $\mathcal{X}$ is one of the varieties $\mathcal{C}$, $\mathcal{SL}$ or $\mathcal{T}$, $\mathcal{G}$ consists of periodic abelian groups and $\mathcal{M}$ satisfies the identities (0.29) and a non-trivial permutation identity of length 4. If the group variety $\mathcal{G}$ is non-trivial then, by Proposition 2.2, the condition 1) of the main theorem holds true. Hence we may assume that $\mathcal{V} = \mathcal{X} \vee \mathcal{M}$. If $\mathcal{X} = \mathcal{T}$ then $\mathcal{V} = \mathcal{M}$ is a nil-variety, and Proposition 2.1 applies. Thus, we may additionally assume that $\mathcal{X}$ is one of the varieties $\mathcal{C}$ or $\mathcal{SL}$. Then $\mathcal{V}$ satisfies the hypothesis of Lemma 2.5 which yields a non-trivial identity (0.30) in the nil-variety $\mathcal{M}$. Since $\mathcal{M}$ satisfies also the identities (0.29), the condition 6) of the main theorem holds for $\mathcal{V} = \mathcal{X} \vee \mathcal{M}$ whenever $\mathcal{X} = \mathcal{C}$. Finally, if $\mathcal{X} = \mathcal{SL}$, then we obtain the condition 5) because, evidently, the combination of the identities (0.29) with a non-trivial identity (0.30) implies one of the identity systems (0.8), (0.14), (0.20) or (0.26). $\square$

It remains to consider the case when $\mathcal{V}$ satisfies the condition c) of Proposition 1.10. This case, however, requires rather cumbersome calculations which we divide into a series of lemmas (Lemmas 2.7–2.12). *All these lemmas deal with a nil-variety $\mathcal{N}$ such that our variety $\mathcal{V}$ can be represented as $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$.*

Lemma 2.7. *Suppose that, for some positive integers m and n with $m \leq n$, every subvariety of $\mathcal{N}$ is (k, m)-split for all k with $m \leq k \leq n$. Then the 0-transversal $W^0_{n,m}(\mathcal{N})$ is a congruence permutable $\mathbb{S}_m$-set.*

Proof. Observe that the hypothesis of the lemma guarantees that $W^0_{n,m}(\mathcal{N})$ is an $\mathbb{S}_m$-set. Take arbitrary congruences α and β on $W^0_{n,m}(\mathcal{N})$ and words $u, v \in W^0_{n,m}(\mathcal{N})$ with $(u, v) \in \alpha\beta$. By symmetry, it is sufficient to verify that $(u, v) \in \beta\alpha$.

There is an element $w \in W^0_{n,m}(\mathcal{N})$ such that $u \ \alpha \ w \ \beta \ v$. By the definition of the set $W^0_{n,m}(\mathcal{N})$, $c(u) = c(w) = c(v) = X_m$. As we have noted before Proposition 1.4, the lattice $\mathrm{Con}(W^0_{n,m}(\mathcal{N}))$ is dually isomorphic to a certain interval of the lattice $L(\mathcal{N})$. Let $\mathcal{A}$ and $\mathcal{B}$ be the subvarieties of $\mathcal{N}$ corresponding to respectively α and β under that dual isomorphism. Further, let $\hat{\alpha}$ and $\hat{\beta}$ be the fully invariant congruences on F corresponding to respectively $\mathcal{A}$ and $\mathcal{B}$. According to the proof of Proposition 1.4 (see [16]), α and β are precisely the restrictions of respectively $\hat{\alpha}$ and $\hat{\beta}$ to $W^0_{n,m}(\mathcal{N})$. Hence $u \ \hat{\alpha} \ w \ \hat{\beta} \ v$, that is, $\mathcal{A}$ satisfies the identity $u = w$ and $\mathcal{B}$ satisfies the identity $w = v$. Put $\mathcal{A}' = \mathcal{A} \vee \mathcal{SL}$ and $\mathcal{B}' = \mathcal{B} \vee \mathcal{SL}$. Since $c(u) = c(w) = c(v)$, Lemma 1.8(i) implies that $u = w = v$ in $\mathcal{SL}$, and therefore, the varieties $\mathcal{A}'$ and $\mathcal{B}'$ satisfy the identities $u = w$ and respectively $w = v$. Let α' and β' be the fully invariant congruences on F corresponding to respectively $\mathcal{A}'$ and $\mathcal{B}'$. Then $u \ \alpha' \ w \ \beta' \ v$, that is, $(u, v) \in \alpha'\beta'$. The congruences α' and β' contain the fully invariant congruence ν corresponding to $\mathcal{V}$ and are contained in the least semilattice congruence on F. Therefore $\alpha'\beta' = \beta'\alpha'$ whence $(u, v) \in \beta'\alpha'$ and $u \ \beta' \ w' \ \alpha' \ v$ for some $w' \in F$. Since $\mathcal{SL} \subseteq \mathcal{A}'$ and $\mathcal{SL} \subseteq \mathcal{B}'$, the identities $u = w' = v$ hold in $\mathcal{SL}$, and therefore, $c(u) = c(w') = c(v)$ by Lemma 1.8(i). In particular, $n(u) = n(w') = n(v) = m$. Furthermore, $u = w'$ in $\mathcal{B}$ and $w' = v$ in $\mathcal{A}$.

It is sufficient to find a word $w'' \in W^0_{n,m}(\mathcal{N})$ such that $u = w''$ in $\mathcal{B}$ and $w'' = v$ in $\mathcal{A}$. Indeed, in this case, $u \ \beta \ w'' \ \alpha \ v$, that is, $(u, v) \in \beta\alpha$ and we are done.

First suppose that $w' \in F_{n,m}(\mathcal{N})$. Put $w'' \equiv (w')^*$. Then $w'' \in W^0_{n,m}(\mathcal{N})$ and $w'' = w'$ in $\mathcal{N}$. Taking into account that $u = w'$ in $\mathcal{B}$, $w' = v$ in $\mathcal{A}$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N}$, we have that $u = w' = w''$ in $\mathcal{B}$ and $w'' = w' = v$ in $\mathcal{A}$, as required.

Now let $w' \notin F_{n,m}(\mathcal{N})$. The definition of the set $F_{n,m}(\mathcal{N})$ and the equalities $c(w') = c(u) = X_m$ imply that either $w' = 0$ in $\mathcal{N}$ or $\ell(w') \neq n$. Put

$w'' \equiv \mathbf{0}$. Clearly, $w'' \in W^0_{n,m}(\mathcal{N})$. It remains to verify that $u = w''$ in $\mathcal{B}$ and $w'' = v$ in $\mathcal{A}$. Note that $w'' = 0$ in $\mathcal{N}$. Hence the identity $w'' = 0$ holds in both the varieties $\mathcal{A}$ and $\mathcal{B}$. Therefore, it suffices to check that $u = 0$ in $\mathcal{B}$ and $v = 0$ in $\mathcal{A}$. If $w' = 0$ in $\mathcal{N}$ then we immediately obtain that $u = w' = 0$ in $\mathcal{B}$ and $v = w' = 0$ in $\mathcal{A}$. Finally, let $\ell(w') \neq n$. Clearly, we may assume that $u, v \in W_{n,m}(\mathcal{N})$. Therefore, $\ell(u) = \ell(v) = n$. Besides that, $n(u) = n(w') = n(v) = m$. Put $k = \min\{n, \ell(w')\}$. Clearly, $m \leq k \leq n$. By the hypothesis, the varieties $\mathcal{A}$ and $\mathcal{B}$ are (k, m)-split. Hence $u = 0$ in $\mathcal{B}$ and $v = 0$ in $\mathcal{A}$. □

Lemma 2.8. *All subvarieties of $\mathcal{N}$ are 3-split.*

Proof. By Lemma 2.5, $\mathcal{N}$ is permutational. In particular, $\mathcal{N}$ is locally nilpotent, whence we may apply Lemma 1.3(iii) to it. Let $\mathcal{Z}$ be a subvariety of $\mathcal{N}$ and $\mathcal{Z}$ satisfies an identity $u = v$ with $\ell(u) = 3$ and $\ell(v) > 3$. We are to check that $u = 0$ in $\mathcal{Z}$. By Lemma 1.3(i), we may assume that $c(u) = c(v)$.

If $n(u) = 1$ or $n(u) = 3$ then it is evident that $u \lhd v$, and Lemma 1.3(iii) applies. It remains to consider the case when $n(u) = 2$. We can assume without any loss that $c(u) = c(v) = \{x, y\}$. Let $k = \ell_x(v)$ and $\ell = \ell_y(v)$. Since $\ell(v) \geq 4$, we have either $k \geq 3$ or $\ell \geq 3$ or $k = \ell = 2$. If $k \geq 3$ or $\ell \geq 3$ then $u \lhd v$ and $u = 0$ in $\mathcal{Z}$ by Lemma 1.3(iii).

Now let $k = \ell = 2$. By Lemma 2.5, $\mathcal{Z}$ satisfies an identity of the kind (0.30) for some non-trivial permutation π. First suppose that $\pi \neq (13)$. It is easy to see that $\mathcal{Z}$ then satisfies an identity of the kind $u = u'$ where $u' \in \{x^2y, y^2x, xy^2, yx^2\}$. On the other hand, by Lemma 1.2, $\mathcal{Z}$ satisfies an identity of the kind $v = v'$ where $v' \in \{x^2y^2, y^2x^2\}$. Since $u' \lhd v'$ and $u' = v'$ in $\mathcal{Z}$, Lemma 1.3(iii) implies $u = u' = 0$ in $\mathcal{Z}$.

Finally let $\pi = (13)$. Since $\ell(u) = 3$, we have $\ell_x(u), \ell_y(u) \leq 2$. Recall that $\ell_x(v) = \ell_y(v) = 2$. Lemma 1.2c) shows that $\mathcal{Z}$ satisfies an identity of the kind $v = v'$ for some word v' such that $\ell(v') = \ell(v) > \ell(u)$ and $u \lhd v'$. By Lemma 1.3(iii), $u = 0$ in $\mathcal{Z}$. □

Lemma 2.9. **a)** *$\mathcal{N}$ satisfies one of the identities*

$$x^2y = 0, \tag{2.2}$$

$$xy^2 = 0, \tag{2.3}$$

$$x^2y = xy^2, \tag{2.4}$$

$$x^2y = yx^2; \tag{2.5}$$

b) *$\mathcal{N}$ satisfies either the identity* (2.2) *or one of the identities*

$$xyx = 0, \tag{2.6}$$

$$x^2y = yxy, \tag{2.7}$$

$$x^2y = xyx. \tag{2.8}$$

Proof. By Lemmas 2.8, 2.7 and 1.5, $W_{3,2}(\mathcal{N})$ is a transitive $\mathbb{S}_2$-set. Suppose that $\mathcal{N}$ satisfies neither (2.2) nor (2.3). Then $x^2y, xy^2 \in F_{3,2}(\mathcal{N})$. We may assume without any loss that $x^2y \in W_{3,2}(\mathcal{N})$. If, besides that, $\mathcal{N}$ satisfies neither (2.4) nor (2.5) then we may also assume that at least one of the words xy^2 and yx^2 lies in $W_{3,2}(\mathcal{N})$. Since $x^2y \not\approx xy^2$ and $x^2y \not\approx yx^2$, we have that $W_{3,2}(\mathcal{N})$ is not transitive, a contradiction. Hence $\mathcal{N}$ satisfies one of the identities (2.2)–(2.5). It proves a), and b) can be verified in a similar way. □

Lemma 2.10. *All subvarieties of* $\mathcal{N}$ *are* $(k,2)$*-split for* $k = 2, 3, 4$.

Proof. For $k = 2$ and $k = 3$, the desired conclusion immediately follows from respectively Corollary 1.7 and Lemma 2.8. Now let $k = 4$. According to Lemma 2.5, $\mathcal{N}$ is permutational, and we may apply Lemma 1.3(iii). Let $\mathcal{Z}$ be a subvariety of $\mathcal{N}$ and $\mathcal{Z}$ satisfies an identity of the kind $u = v$ with $\ell(u) = 4$, $n(u) = 2$ and $\ell(v) > 4$. We have to verify that $u = 0$ in $\mathcal{Z}$. By Lemma 1.3(i), we can assume that $c(u) = c(v)$.

By Lemma 2.5, $\mathcal{N}$ satisfies a non-trivial identity of the kind (0.30). Hence, by Lemma 1.2, $\mathcal{Z}$ satisfies an identity of the kind $u = u'$ where u' is similar to one of the words x^3y, xy^3 or x^2y^2. Hence we may assume that $u \in \{x^3y, xy^3, x^2y^2\}$, and therefore, $c(v) = \{x, y\}$. Put $k = \ell_x(v)$ and $\ell = \ell_y(v)$. If $k \geq 4$ or $\ell \geq 4$ then $u \lhd v$ and $u = 0$ in $\mathcal{Z}$ by Lemma 1.3(iii). Since $\ell(v) \geq 5$, we have either $k = \ell = 3$ or $k = 3$, $\ell = 2$ or $k = 2$, $\ell = 3$.

By Lemma 1.2, v equals in $\mathcal{Z}$ to one of the words x^3y^3, y^3x^3, x^3y^2, y^3x^2, x^2y^3 or y^2x^3. Hence we may assume that v is one of these six words. Lemma 1.3(iii) immediately implies $u = 0$ in $\mathcal{Z}$ whenever either $u \equiv x^2y^2$ or $v \in \{x^3y^3, y^3x^3\}$ or $u \equiv x^3y$, $v \in \{x^3y^2, y^3x^2\}$ or $u \equiv xy^3$, $v \in \{x^2y^3, y^2x^3\}$. The two following possibilities remain: either $u \equiv x^3y$, $v \in \{x^2y^3, y^2x^3\}$ or $u \equiv xy^3$, $v \in \{x^3y^2, y^3x^2\}$.

By Lemma 2.9a), $\mathcal{N}$ satisfies one of the identities (2.2)–(2.5). Each of the identities (2.2) and (2.3) immediately implies $u = 0$ in $\mathcal{Z}$. Suppose that $\mathcal{N}$ satisfies (2.4). Then $x^3y = x \cdot x^2y = x^2y^2 = x^4y$ and $xy^3 = xy^2 \cdot y = x^2y^2 = xy^4$ in $\mathcal{Z}$. Thus, $\mathcal{Z}$ satisfies an identity of the kind $u = u'$ for some word u' with $u \lhd u'$. By Lemma 1.3(iii), $u = 0$ in $\mathcal{Z}$ in this case. Finally, suppose that $\mathcal{N}$ satisfies (2.5). Then either $u \equiv x^3y$ and $v = v'$ in $\mathcal{Z}$ where $v' \in \{y^3x^2, x^3y^2\}$ or $u \equiv xy^3$ and $v = v'$ in $\mathcal{Z}$ where $v' \in \{y^2x^3, x^2y^3\}$. In both cases, $u \lhd v'$, and Lemma 1.3(iii) applies again. □

Lemma 2.11. $\mathcal{N}$ *satisfies one of the identities*

(2.9) $$x^3y = 0,$$

(2.10) $$x^2y^2 = 0,$$

(2.11) $$x^3y = x^2y^2.$$

Proof. By Lemmas 2.10, 2.7 and 1.5, $W_{4,2}(\mathcal{N})$ is a transitive $\mathbb{S}_2$-set. Suppose that $\mathcal{N}$ satisfies neither (2.9) nor (2.10). Then $x^3y, x^2y^2 \in F_{4,2}(\mathcal{N})$. We may assume without any loss that $x^3y \in W_{4,2}(\mathcal{N})$. If, besides that, $\mathcal{N}$ satisfies neither (2.11) nor the identity

$$x^3y = y^2x^2 \tag{2.12}$$

then we may also assume that at least one of the words x^2y^2 and y^2x^2 lies in $W_{4,2}(\mathcal{N})$. Since $x^3y \not\approx x^2y^2$ and $x^3y \not\approx y^2x^2$, we have that $W_{4,2}(\mathcal{N})$ is not transitive, a contradiction. Hence $\mathcal{N}$ satisfies one of the identities (2.9)–(2.12). If one of the identities (2.9)–(2.11) holds in $\mathcal{N}$ then we are done.

It remains to consider the case when $\mathcal{N}$ satisfies (2.12). By Lemma 2.9a), one of the identities (2.2)–(2.5) holds in $\mathcal{N}$. Clearly, each of the identities (2.2), (2.3) and (2.5) together with (2.12) imply (2.11). Finally, suppose that (2.4) holds in $\mathcal{N}$. Substituting y^2 for y in this identity, we obtain $x^2y^2 = xy^4$. By Lemma 2.5, $\mathcal{N}$ is permutational, and we may use Lemma 1.3(iii). It implies that $\mathcal{N}$ satisfies $x^2y^2 = 0$. Clearly, this identity and (2.12) imply (2.11). □

Lemma 2.12. **a)** *If $\mathcal{N}$ satisfies one of the identities* (2.4) *or* (2.7) *then it also satisfies $x^2yz = 0$.*

b) *If $\mathcal{N}$ satisfies the identity* (2.11) *then it also satisfies $x^2y^2z = 0$.*

Proof. By Lemma 2.5 $\mathcal{N}$ satisfies a non-trivial identity of the form (0.30). Suppose that $\mathcal{N}$ satisfies the identity (2.4). Multiplying this identity by z on the right, we get $x^2yz = xy^2z$. On the other hand, substituting yz for y in (2.4) and using Lemma 1.2, we obtain $x^2yz = xy^2z^2$. Therefore, $xy^2z = xy^2z^2$ in $\mathcal{N}$ and by Lemma 1.3(iii) $\mathcal{N}$ satisfies the identities $x^2yz = xy^2z = 0$.

The cases when $\mathcal{N}$ satisfies one of the identities (2.7) or (2.11) can be verified in a similar way. □

Proposition 2.13. *If $\mathcal{V}$ satisfies the condition* c) *of Proposition* 1.10*, then $\mathcal{V}$ satisfies one of the conditions* 4) *or* 5) *of the main theorem.*

Proof. Recall that the condition c) means that $\mathcal{V} = \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{Y}$ is one of the varieties $\mathcal{SL}$ or $\mathcal{T}$ and $\mathcal{N}$ is a nil-variety and satisfies a permutation identity of length 4 for some non-trivial even permutation. If $\mathcal{Y} = \mathcal{T}$, then $\mathcal{V} = \mathcal{N}$ is a nil-variety, and by Proposition 2.1, it satisfies the condition 4). Thus, we may assume that $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$. We are going to verify that the condition 5) holds, that is, $\mathcal{N}$ satisfies one of the identity systems (0.8)–(0.28). According to Lemma 2.5, $\mathcal{N}$ satisfies one of the identities $xyz = yxz$, $xyz = xzy$, $xyz = zyx$ or $xyz = yzx$. Consider the four corresponding cases.

Case 1: *$\mathcal{N}$ satisfies the identity $xyz = yxz$.* By Lemma 2.9a), one of the identities (2.2)–(2.5) holds in $\mathcal{N}$. If $\mathcal{N}$ satisfies one of the identities (2.2) or (2.3) then we immediately get one of the identity systems (0.8) or (0.9), respectively. If the identity (2.4) holds in $\mathcal{N}$ then, in view of Lemma 2.12a), we get the identity system (0.10). Finally, let $\mathcal{N}$ satisfy the identity (2.5). By Lemma 2.11, one of the identities (2.9)–(2.11) holds in $\mathcal{N}$. If $\mathcal{N}$ satisfies one of the identities (2.9) or (2.10) we obtain one of the identity systems (0.11) or

(0.12), respectively. In the case when (2.11) holds in $\mathcal{N}$, by Lemma 2.12b), we have the identity system (0.13).

Case 2: *$\mathcal{N}$ satisfies the identity $xyz = xzy$.* The same arguments as in Case 1 show that $\mathcal{N}$ satisfies one of the identity systems (0.14)–(0.19).

Case 3: *$\mathcal{N}$ satisfies the identity $xyz = zyx$.* Repeating the reasoning from Case 1 but referring to Lemma 2.9b) rather than Lemma 2.9a), we conclude that $\mathcal{N}$ satisfies one of the identity systems (0.20)–(0.25).

Case 4: *$\mathcal{N}$ satisfies the identity $xyz = yzx$.* By Lemma 2.11, one of the identities (2.9)–(2.11) holds in $\mathcal{N}$. If $\mathcal{N}$ satisfies one of the identities (2.9) or (2.10), then we immediately get one of the identity systems (0.26) or (0.27), respectively. In the case when (2.11) holds in $\mathcal{N}$, by Lemma 2.12b), we obtain the identity system (0.28). □

The "only if" part of the main theorem immediately follows from Lemma 2.3 and Propositions 1.10, 2.4, 2.6 and 2.13.

3. Sufficiency

We are going to verify that if a semigroup variety $\mathcal{V}$ satisfies one of the conditions 1)–6) of the main theorem, then the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups commute. In fact, completely regular semigroup varieties, that is, varieties satisfying 1), are known to enjoy a stronger property: the sub-semilattice fully invariant congruences on every relatively free completely regular semigroup commute [8, 9]. The same property holds for relatively free members of varieties satisfying 2), that is, the varieties $\mathcal{P}$ and $\mathcal{P}^*$, [15, Lemma 2.6]. For varieties satisfying 4), [15, Proposition 1.15] yields a property which is even stronger: all fully invariant congruences on their relatively free members commute; moreover, the product of any two such congruences coincides with their set-theoretical union. Thus, it remains to consider the conditions 3), 5) and 6).

Suppose that $\mathcal{V}$ satisfies 3), that is, $\mathcal{V}$ contains the variety $\mathcal{SL}$ and satisfies the identity (0.1). We employ a technique similar to that of [8, 9], see also [21]. Let ν and σ be the fully invariant congruences on F corresponding to the varieties $\mathcal{SI} = \mathrm{var}\{(xy)^2 = xy\}$ and $\mathcal{SL}$, respectively. Clearly, $\nu \subset \sigma$.

Lemma 3.1 ([21, Proposition 1.8]). *Let $u, v \in F$. Then $u^\nu \,\mathbf{R}\, v^\nu$ in F/ν if and only if either u and v coincide with the same letter or $\ell(u) > 1$, $\ell(v) > 1$ and u and v have the same left indicator.* □

Given a congruence μ on F, we define, following [10], a new relation μ_0 on F by letting $u\ \mu_0\ v$ if and only if there are $u', v' \in F$ with $u \equiv s(u')$, $v \equiv s(v')$ and $u'\ \mu\ v'$. We denote by λ the fully invariant congruence on F corresponding to the variety $\mathcal{LRB} = \mathrm{var}\{x^2 = x,\ xyx = xy\}$. Clearly, $\nu \subset \lambda \subset \sigma$.

Lemma 3.2 ([21, Propositions 2.3 and 2.4])**.** *Let μ be a fully invariant congruence on F that belongs to the interval $[\nu, \lambda]$. Then*

a) *μ_0 is a fully invariant congruence that belongs to the interval $[\nu, \lambda]$;*
b) *for any $u, v \in F$, $u^\mu \, \mathbf{R} \, v^\mu$ in F/μ whenever $u \; \mu_0 \; v$.* □

Proposition 3.3. *Let a semigroup variety $\mathcal{V}$ satisfy the condition* 3) *of the main theorem. Then the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups commute.*

Proof. By Lemma 1.11 it suffices to show that $\alpha\beta = \beta\alpha$ for any fully invariant congruences α, β on F such that $\alpha, \beta \in [\nu, \sigma]$. We will verify that $\alpha\beta \subseteq \beta\alpha$, the claim will then follow by symmetry. Let $(u, v) \in \alpha\beta$, that is, $u \; \alpha \; w \; \beta \; v$ for some word $w \in F$. By Lemma 1.8(i), $c(u) = c(w) = c(v)$; in particular, $n(u) = n(w) = n(v)$. We induct on $n(u)$ to show that $(u, v) \in \beta\alpha$.

First suppose that $n(u) = 1$ and $c(u) = c(w) = c(v) = \{x\}$. Then $u \equiv x^p$, $w \equiv x^q$ and $v \equiv x^r$ for some positive integers p, q and r. Substituting x for y in the identity (0.1), we get $x^2 = x^4$, whence $x^2 = x^6$. On the other hand, substituting x^2 for y in (0.1) yields $x^3 = x^6$. Therefore, $x^2 = x^3 = \ldots$ in $\mathcal{SI}$, and we may assume that $p, q, r \le 2$. Then some of these numbers must be equal by the pigeon-hole principle. If $q = r$ or $p = r$, then $u \; \alpha \; v$, and if $p = q$, then $u \; \beta \; v$. In both the cases $(u, v) \in \beta\alpha$.

Now suppose that the words u, v and w depend on ≥ 2 letters. We first prove that there is a word a with the following properties:

$$u^\beta \, \mathbf{R} \, a^\beta \text{ in } F/\beta, \quad a^\alpha \, \mathbf{R} \, v^\alpha \text{ in } F/\alpha \quad \text{and} \quad c(a) = c(u). \tag{3.1}$$

According to [21, the proof of Theorem 3.1], if at least one of the congruences α and β is not contained in λ then either $u^\alpha \, \mathbf{R} \, v^\alpha$ in F/α or $u^\beta \, \mathbf{R} \, v^\beta$ in F/β. In both the cases (3.1) holds true (we may let $a \equiv u$ in the former case and $a \equiv v$ in the latter one).

Now let $\alpha, \beta \subseteq \lambda$. Recall that $u \; \alpha \; w \; \beta \; v$. Hence $s(u) \; \alpha_0 \; s(w) \; \beta_0 \; s(v)$, that is, $(s(u), s(v)) \in \alpha_0\beta_0$. By Lemma 3.2a), α_0 and β_0 are fully invariant congruences from the interval $[\nu, \lambda] \subseteq [\nu, \sigma]$. Lemma 1.8(i) then implies that $c(s(u)) = c(s(w)) = c(s(v))$. This allows us to apply the induction assumption to the congruences α_0, β_0 and the words $s(u), s(v)$ containing less letters than u and v. Therefore, $(s(u), s(v)) \in \beta_0\alpha_0$, that is, there exists a word f such that $s(u) \; \beta_0 \; f \; \alpha_0 \; s(v)$. Furthermore, the equalities $c(u) = c(v)$ and $c(s(u)) = c(s(v))$ imply that $\tau(u) \equiv \tau(v)$. Put $x \equiv \tau(u)$. Then $s(u)x \; \beta_0 \; fx \; \alpha_0 \; s(v)x$. By Lemma 3.2b),

$$(s(u)x)^\beta \, \mathbf{R} \, (fx)^\beta \text{ in } F/\beta \quad \text{and} \quad (fx)^\alpha \, \mathbf{R} \, (s(v)x)^\alpha \text{ in } F/\alpha.$$

It is clear that the words u and $s(u)x$ (respectively v and $s(v)x$) have the same left indicator. By Lemma 3.1, this implies that $(s(u)x)^\nu \, \mathbf{R} \, u^\nu$ and $(s(v)x)^\nu \, \mathbf{R} \, v^\nu$ in F/ν. One can note that Lemmas 1.8(i) and 3.2a) imply that $c(f) = c(s(u))$, and therefore, $c(fx) = c(u)$. Hence (3.1) holds with $a \equiv fx$.

By symmetry, one can verify that there is a word b such that

$$u^\beta \,\mathbf{L}\, b^\beta \text{ in } F/\beta, \quad b^\alpha \,\mathbf{L}\, v^\alpha \text{ in } F/\alpha \quad \text{and} \quad c(b) = c(u). \tag{3.2}$$

Since $\ell(u), \ell(v) \geq 2$, the identity (0.1) implies that u^ν and v^ν are idempotents of the semigroup F/ν. Using (3.1) and (3.2) together with the fact that idempotents are left identities of their **R**-classes and right identities of their **L**-classes [1, Lemma 2.14], we obtain $au \;\beta\; u \;\beta\; ub$. Multiplying the first pair on the right by b, we get $aub \;\beta\; ub$, whence $u \;\beta\; aub$. Analogously, $av \;\alpha\; v \;\alpha\; vb$, whence $v \;\alpha\; avb$. Clearly, the words aub and avb have the same left indicator. By Lemma 3.1, $(aub)^\nu \,\mathbf{R}\, (avb)^\nu$ in F/ν. By symmetry, $(aub)^\nu \,\mathbf{L}\, (avb)^\nu$ in F/ν whence $(aub)^\nu \,\mathbf{H}\, (avb)^\nu$ in F/ν. Thus $(aub)^\nu$ and $(avb)^\nu$ belong to the same **H**-class of F/ν being a subgroup of F/ν. The identity (0.1) forces all subgroups in F/ν to be singletons. Therefore, $aub \;\nu\; avb$. Hence $u \;\beta\; aub \;\nu\; avb \;\alpha\; v$. Since $\nu \subseteq \beta$, we have $(u, v) \in \beta\alpha$, as required. □

The following observation is crucial for analyzing the conditions 5) and 6):

Lemma 3.4. *Let $\mathcal{N}$ be the variety defined by one of the identity systems* (0.8)–(0.28)*. Then $\mathcal{N}$ is hereditarily homogeneous and, for any pair of positive integers m and n with $m \leq n$ and for any subvariety $\mathcal{M} \subseteq \mathcal{N}$, the congruences of the $\mathbb{S}_m$-set $(W^0_{n,m}(\mathcal{M}))$ form a chain.*

Proof. In Table 1 (see the next page) we list all possible types of non-empty transversals $W_{n,m}(\mathcal{N})$ with $1 < m < n$ and, for each of them, we list (up to similarity) all words from that transversal. The facts collected in Table 1 can be verified by means of straightforward, but lengthy calculations based on Lemmas 1.3 and 1.2. We allow ourselves to omit these calculations.

In order to verify that $\mathcal{N}$ is hereditarily homogeneous, we take an arbitrary subvariety $\mathcal{M} \subseteq \mathcal{N}$ and suppose that $\mathcal{M}$ satisfies an identity $u = v$ with $\ell(u) < \ell(v)$. We have to show that $u = 0$ in $\mathcal{M}$. Clearly, we may assume that $u, v \neq 0$ in $\mathcal{N}$. By Lemma 1.3(i) we may also assume that $c(u) = c(v)$. Since $\mathcal{N}$ is permutational, and therefore, locally nilpotent, Lemma 1.3(iii) applies. Thus, it suffices to verify that $u \lhd v$. This is evident whenever $n(u) = 1$ or $n(u) = \ell(u)$. If $1 < n(u) < \ell(u)$, then the words u and v should be similar to one of the words in the right-hand column of Table 1. An immediate inspection shows that the conditions $c(u) = c(v)$ and $\ell(u) < \ell(v)$ always ensure that $u \lhd v$ in this case.

It remains to verify that the lattice $\mathrm{Con}(W^0_{n,m}(\mathcal{M}))$ is a chain. From Table 1 we see that, if $1 < m < n$ and $W_{n,m}(\mathcal{M}) \neq \varnothing$, then $u \approx v$ for all $u, v \in W_{n,m}(\mathcal{M})$. Thus, all these transversals are transitive. Clearly, non-empty transversals of the form $W_{n,1}(\mathcal{M})$ or $W_{n,n}(\mathcal{M})$ are always transitive. Therefore all 0-transversals $W^0_{n,m}(\mathcal{M})$ have at most 2 orbits: the singleton

Table 1. Non-empty transversals.

$\mathcal{N} = \operatorname{var}\Sigma$ where Σ is one of the identity systems	Non-empty transversals $W_{n,m}(\mathcal{N})$ with $1 < m < n$	Every word from $W_{n,m}(\mathcal{N})$ is similar to the word
(0.8)	$W_{m+1,m}(\mathcal{N})$	$x_1x_2\cdots x_{m-1}x_m^2$
(0.9), (0.10), (0.16), (0.22)	$W_{3,2}(\mathcal{N})$	x^2y
(0.11), (0.17), (0.23), (0.26)	$W_{m+k,m}(\mathcal{N})$, $k < m$	$x_1^2x_2^2\cdots x_k^2x_{k+1}\cdots x_m$
	$W_{2m,m}(\mathcal{N})$	$x_1^2x_2^2\cdots x_m^2$
(0.12), (0.18), (0.24), (0.27)	$W_{m+1,m}(\mathcal{N})$	$x_1^2x_2\cdots x_m$
	$W_{m+2,m}(\mathcal{N})$	$x_1^3x_2\cdots x_m$
(0.13), (0.19), (0.25), (0.28)	$W_{4,2}(\mathcal{N})$	x^3y
	$W_{m+1,m}(\mathcal{N})$	$x_1^2x_2\cdots x_m$
(0.14), (0.21)	$W_{3,2}(\mathcal{N})$	xy^2
(0.15)	$W_{m+1,m}(\mathcal{N})$	$x_1^2x_2\cdots x_m$
(0.20)	$W_{3,2}(\mathcal{N})$	xyx

orbit $\{\mathbf{0}\}$ and $W_{n,m}(\mathcal{M})$ provided that the latter is non-empty. Then the lattice $\operatorname{Con}(W_{n,m}^0(\mathcal{M}))$ either has only one element or is isomorphic to the lattice $\operatorname{Con}(W_{n,m}(\mathcal{M}))$ with the new greatest element adjoined. We see that it suffices to show that the latter lattice is a chain whenever $W_{n,m}(\mathcal{M}) \neq \varnothing$ (what will be assumed for the rest of the proof). In fact, we will prove more: $W_{n,m}(\mathcal{M})$ has at most 3 congruences.

By Lemma 1.6, the lattice $\operatorname{Con}(W_{n,m}(\mathcal{M}))$ is isomorphic to an interval in the lattice $\operatorname{Sub}(\mathbb{S}_m)$. If $m \le 2$ then the whole lattice $\operatorname{Sub}(\mathbb{S}_m)$ contains at most 2 elements. Hence we may assume that $m > 2$.

First consider the case $m = n$. By Corollary 1.7, the lattice $\operatorname{Con}(W_{n,n}(\mathcal{M}))$ is isomorphic to the interval $[\operatorname{Perm}_n(\mathcal{M}), \mathbb{S}_n]$. Since each of the identity systems (0.8)–(0.28) includes a non-trivial permutation identity of length 3, the subgroup $\operatorname{Perm}_3(\mathcal{M})$ contains a non-trivial permutation. As every non-singleton subgroup is maximal in $\mathbb{S}_3$, the interval $[\operatorname{Perm}_3(\mathcal{M}), \mathbb{S}_3]$ contains at most 2 elements. If $n \ge 4$ then, by Lemma 1.2, $\operatorname{Perm}_n(\mathcal{M})$ contains one of the groups $\operatorname{Stab}_n(1)$ or $\operatorname{Stab}_n(n)$. Since both of these groups are maximum proper subgroups of $\mathbb{S}_n$, we have that $\operatorname{Con}(W_{n,n}(\mathcal{M}))$ with $n \ge 4$ contains at most 2 elements as well.

Now let $m < n$. An inspection of Table 1 shows that we have to analyze the following transversals:

a) $W_{m+1,m}(\mathcal{M})$ if $\mathcal{M}$ satisfies one of the systems (0.8), (0.12), (0.13), (0.15), (0.18), (0.19), (0.24), (0.25), (0.27) or (0.28);
b) $W_{m+2,m}(\mathcal{M})$ if $\mathcal{M}$ satisfies one of the systems (0.12), (0.18), (0.24) or (0.27);

c) $W_{m+k,m}(\mathcal{M})$ for all $k < m$ if $\mathcal{M}$ satisfies one of the systems (0.11), (0.17), (0.23) or (0.26);
d) $W_{2m,m}(\mathcal{M})$ if $\mathcal{M}$ satisfies one of the systems (0.11), (0.17), (0.23) or (0.26).

Let W be one of these transversals and $u \in W$. Since $2 < m < n$, $\ell(u) > 3$. Using Lemma 1.2, we easily calculate $\mathrm{Stab}_W(u)$ in each of the cases a)–d). Namely, we have respectively one of the following possibilities:

a) $\mathrm{Stab}_W(u) \supseteq \mathrm{Stab}_m(\ell)$, where $\ell = \begin{cases} m & \text{if } \mathcal{M} \text{ satisfies (0.8)}, \\ 1 & \text{otherwise}; \end{cases}$
b) $\mathrm{Stab}_W(u) \supseteq \mathrm{Stab}_m(1)$;
c) $\mathrm{Stab}_W(u) \supseteq G_{m,k}$;
d) $\mathrm{Stab}_W(u) = \mathbb{S}_m$.

Therefore $\mathrm{Con}(W)$ has at most 2 elements in the cases a) and b), at most 3 elements in the case c) (by Lemma 1.1), and 1 element in the case d). □

Proposition 3.5. *Let a semigroup variety $\mathcal{V}$ satisfy the condition* 5) *of the main theorem. Then the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups commute. Moreover, the product of any two such congruences coincides with their set-theoretical union.*

Proof. Recall that 5) means that $\mathcal{V} = \mathcal{SL} \vee \mathcal{M}$ where $\mathcal{M}$ satisfies one of the identity systems (0.8)–(0.28). We denote by ν, μ and σ the fully invariant congruences on F corresponding to respectively $\mathcal{V}$, $\mathcal{M}$ and $\mathcal{SL}$. By Lemma 1.11 it suffices to verify that $\alpha_1\alpha_2 = \alpha_1 \cup \alpha_2$ for any fully invariant congruences α_1, α_2 on F such that $\alpha_1, \alpha_2 \in [\nu, \sigma]$. We denote by $\mathcal{A}_1$ and $\mathcal{A}_2$ the subvarieties of $\mathcal{V}$ corresponding to respectively α_1 and α_2. It is known that the mapping $\mathcal{A} \mapsto \mathcal{A} \vee \mathcal{SL}$ is an isomorphism between $L(\mathcal{M})$ and $[\mathcal{SL}, \mathcal{V}]$ (cf. [7]). Hence $\mathcal{A}_i = \mathcal{SL} \vee \mathcal{M}_i$ for some subvariety $\mathcal{M}_i$ of $\mathcal{M}$, $i = 1, 2$. Let μ_1 and μ_2 be the fully invariant congruences on F corresponding to respectively $\mathcal{M}_1$ and $\mathcal{M}_2$. Clearly, $\alpha_i = \mu_i \wedge \sigma$ for $i = 1, 2$.

Let $(u, v) \in \alpha_1\alpha_2$, that is, $u \; \alpha_1 \; w \; \alpha_2 \; v$ for some word $w \in F$. Clearly, $u \; \mu_1 \; w \; \mu_2 \; v$ and $u \; \sigma \; w \; \sigma \; v$. By Lemma 1.8(i), $c(u) = c(w) = c(v)$. Put $m = n(u)$. We may assume without any loss that $c(u) = X_m$. Since $u \; \sigma \; v$, it suffices to verify that either $u \; \mu_1 \; v$ or $u \; \mu_2 \; v$.

If $\mathcal{M}$ satisfies $u = 0$ as well as $v = 0$ then $u \; \mu_1 \; v$. Hence we may also assume that $u \neq 0$ in $\mathcal{M}$. Consider the four following cases:

Case 1: $\ell(u) = \ell(w) = \ell(v)$. Put $n = \ell(u)$ and $W = W^0_{n,m}(\mathcal{M})$. Then there are $u_1, w_1, v_1 \in W$ such that $\mathcal{M}$ satisfies the identities $u = u_1$, $w = w_1$ and $v = v_1$. Put $\mu_i' = \mu_i|_W$, $i = 1, 2$. It is shown in [16] (and easy to verify) that μ_1' and μ_2' are congruences on the $\mathbb{S}_m$-set W. We have

$$u_1 \; \mu \; u \; \mu_1 \; w \; \mu \; w_1 \; \mu \; w \; \mu_2 \; v \; \mu \; v_1,$$

whence $u_1\ \mu_1'\ w_1\ \mu_2'\ v_1$. By Lemma 3.4, the congruences μ_1' and μ_2' are comparable. Therefore, if $\mu_2' \subseteq \mu_1'$, then $u\ \mu\ u_1\ \mu_1'\ v_1\ \mu\ v$ and $u\ \mu_1\ v$, and if $\mu_1' \subseteq \mu_2'$, then $u\ \mu_2\ v$.

Case 2: $\ell(u) = \ell(w) \neq \ell(v)$. By Lemma 3.4, the variety $\mathcal{M}$ is hereditarily homogeneous, whence $w = v = 0$ in $\mathcal{M}_2$. Suppose that $w \neq 0$ in $\mathcal{M}$. Put $n = \ell(u)$ and $W = W_{n,m}(\mathcal{M})$. Since $u, w \neq 0$ in $\mathcal{M}$, we may consider the words u^* and w^*. We note that $\mathcal{M}$ satisfies the identities $u = u^*$ and $w = w^*$. An inspection of Table 1 shows that W is transitive. Hence $u^* \equiv \xi^*(w^*)$ for some permutation $\xi \in \mathbb{S}_m$. We have

$$u\ \mu\ u^* \equiv \xi^*(w^*) \equiv (w^*\xi)^*\ \mu\ w^*\xi\ \mu\ w\xi.$$

Since $\mu \subseteq \mu_2$, we obtain $u = w\xi = 0$ in $\mathcal{M}_2$, whence $u\ \mu_2\ v$.

Now let $w = 0$ in $\mathcal{M}$. Clearly, $u = 0$ in $\mathcal{M}_1$ in this case. If, besides that, $v = 0$ in $\mathcal{M}$ then $u\ \mu_1\ v$. Suppose that $v \neq 0$ in $\mathcal{M}$. Recall that $u \neq 0$ in $\mathcal{M}$, $c(u) = c(v)$ and $\ell(u) \neq \ell(v)$. Again, looking at Table 1, we may see that either $u \lhd v$ or $v \lhd u$. In the former case, $v = 0$ in $\mathcal{M}_1$ whence $u\ \mu_1\ v$, and in the latter one $u = 0$ in $\mathcal{M}_2$ whence $u\ \mu_2\ v$.

Case 3: $\ell(u) \neq \ell(w) = \ell(v)$. This case is dual to the previous one.

Case 4: $\ell(u) \neq \ell(w) \neq \ell(v)$. In this case, $u = 0$ in $\mathcal{M}_1$ and $v = 0$ in $\mathcal{M}_2$. Hence we can repeat the argument from the second paragraph of Case 2. □

Proposition 3.6. *Let a semigroup variety $\mathcal{V}$ satisfy the condition* 6) *of the main theorem. Then the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups commute. Moreover, the product of any two such congruences coincides with their set-theoretical union.*

Proof. Recall that 6) means that $\mathcal{V} = \mathcal{C} \vee \mathcal{N}$ where $\mathcal{N}$ satisfies the identities (0.29) and a non-trivial identity (0.30). We denote by ν, μ and σ the fully invariant congruences on F corresponding to respectively $\mathcal{V}$, $\mathcal{N}$ and $\mathcal{SL}$. By Lemma 1.11 it suffices to verify that $\alpha_1\alpha_2 = \alpha_1 \cup \alpha_2$ for any fully invariant congruences α_1, α_2 on F such that $\alpha_1, \alpha_2 \in [\nu, \sigma]$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the subvarieties of $\mathcal{V}$ corresponding to respectively α_1 and α_2. By [17, Lemma 2], $\mathcal{A}_i = \mathcal{X}_i \vee \mathcal{N}_i$ where $\mathcal{X}_i$ is one of the varieties $\mathcal{C}$ or $\mathcal{SL}$ and $\mathcal{N}_i \subseteq \mathcal{N}$, $i = 1, 2$.

First suppose that $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{SL}$. Let $\mathcal{V}' = \mathcal{SL} \vee \mathcal{N}$. Obviously, $\mathcal{N}$ satisfies one of the identity systems (0.8), (0.14), (0.20) or (0.26), whence $\mathcal{V}'$ satisfies the condition 5) of the main theorem. Let ν' be the fully invariant congruence on F corresponding to $\mathcal{V}'$. It is clear that $\alpha_1, \alpha_2 \supseteq \nu'$. By Proposition 3.5 and Lemma 1.11 we then obtain that $\alpha_1\alpha_2 = \alpha_1 \cup \alpha_2$.

Thus, we may assume without any loss that $\mathcal{X}_1 = \mathcal{C}$. In particular, $\mathcal{X}_1 \supseteq \mathcal{X}_2$. Suppose that $(u, v) \in \alpha_1\alpha_2$, that is, $u\ \alpha_1\ w\ \alpha_2\ v$ for some word $w \in F$. Thus, $u = w$ in $\mathcal{A}_1$ and $w = v$ in $\mathcal{A}_2$. In particular, $u = w = v$ in $\mathcal{SL}$, and by Lemma 1.8(i), $c(u) = c(w) = c(v)$. We want to show that either $u\ \alpha_1\ v$ or $u\ \alpha_2\ v$. Clearly, we may also assume that the words u, w, v are pairwise different: otherwise the desired conclusion follows immediately.

Observe that if $u = w$ in $\mathcal{N}_2$ (in particular, if $u = w$ in $\mathcal{N}$), then we are done. Indeed, the identity $u = w$ is satisfied by $\mathcal{X}_2$ because it holds in $\mathcal{X}_1$ and $\mathcal{X}_1 \supseteq \mathcal{X}_2$. Hence $u = w$ in $\mathcal{A}_2 = \mathcal{X}_2 \vee \mathcal{N}_2$. Therefore, $u = w = v$ in $\mathcal{A}_2$, that is, $u \; \alpha_2 \; v$.

Now consider two possibilities.

Case 1: *the word u is linear.* Since the identity $u = w$ holds in $\mathcal{C}$, Lemma 1.8(iv) applies forcing $u = w$ to be a permutation identity. Let ξ be the corresponding permutation. Since $u \not\equiv w$, we have $n(u) = n(w) > 1$. Put $n = n(u)$. Clearly, $\xi \in \mathrm{Perm}_n(\mathcal{N}_1)$. According to the remark above, we assume that $u = w$ fails in $\mathcal{N}$, whence $\xi \notin \mathrm{Perm}_n(\mathcal{N})$. We intend to show that $\mathrm{Perm}_n(\mathcal{N}_1) = \mathbb{S}_n$. If $n = 2$ then it follows from the fact that the group $\mathrm{Perm}_n(\mathcal{N}_1)$ is non-trivial. If $n \geq 3$ then the fact that $\mathcal{N}$ satisfies a non-trivial identity (0.30) and Lemma 1.2 imply that $\mathrm{Perm}_n(\mathcal{N})$ is a maximum proper subgroup of $\mathbb{S}_n$. Therefore $\mathrm{Perm}_n(\mathcal{N}_1)$ containing both $\mathrm{Perm}_n(\mathcal{N})$ and ξ must be equal to $\mathbb{S}_n$. Thus, $\mathcal{N}_1$ satisfies all permutation identities of length n, and so does $\mathcal{A}_1$.

If v is a linear word then $u = v$ is a permutation identity of length n, whence $u \; \alpha_1 \; v$. If v is not a linear word, then $\ell(v) \neq n$. By Lemma 1.3(ii), $\mathcal{N}_2$ satisfies $x_1 x_2 \cdots x_n = 0$, whence it satisfies all permutation identities of length n. In particular, $u = w$ holds in $\mathcal{N}_2$. As we noted above, $u \; \alpha_2 \; v$ under this condition.

Case 2: *the word u is not linear.* First suppose that $n(u) > 1$. Since the identity $u = w$ holds in the variety $\mathcal{C}$, Lemma 1.8(iv) shows that the word w is not linear and $n(w) > 1$. Then the identity $u = w$ follows from the identities (0.29). Hence $u = w$ in $\mathcal{N}$ and $u \; \alpha_2 \; v$, as we saw above.

Now let $n(u) = 1$, that is, $u \equiv x^k$ for some $k > 1$. In this case, $w \equiv x^m$ and $v \equiv x^n$ for some positive integers m and n. Clearly, k, m and n are pairwise different. Since $u = w$ in $\mathcal{C}$, Lemma 1.8(iv) implies that $m > 1$. If $k, m \geq 3$ then the identity $x^k = x^m$ follows from the identities (0.29), and therefore, $u = w$ holds in $\mathcal{N}$. As we noted above, it ensures $u \; \alpha_2 \; v$. Thus, we may assume that either $k = 2, m > 2$ or $k > 2, m = 2$. Suppose that $n = 1$. Since the identity $w = v$ holds in $\mathcal{N}_2$, this variety is trivial by Lemma 1.3(iii). Hence $u = w$ in $\mathcal{N}_2$, and therefore, $u \; \alpha_2 \; v$. Let $n > 1$. Then $n > 2$ because $k \neq n$ and $m \neq n$. Since the identity $u = w$ holds in $\mathcal{N}_1$, this variety satisfies the identity $x^2 = 0$ by Lemma 1.3(iii). It means that $\mathcal{N}_1$ satisfies either $u = v$ (if $k = 2$, $m > 2$) or $w = v$ (if $k > 2$, $m = 2$). The identities $u = v$ and $w = v$ both hold in $\mathcal{C}$. If $\mathcal{N}_1$ satisfies the identity $u = v$, then $u = v$ in $\mathcal{A}_1$, that is, $u \; \alpha_1 \; v$. Finally, if $\mathcal{N}_1$ satisfies the identity $w = v$, then $\mathcal{A}_1 = \mathcal{C} \vee \mathcal{N}_1$ satisfies $u = w = v$, and therefore, $u \; \alpha_1 \; v$ as well. □

As discussed at the beginning of this section, the "if" part of the main theorem follows from Propositions 3.3, 3.5 and 3.6.

4. A relationship with the distributive law in subvariety lattices

As mentioned in the introduction, the proof of the main theorem reveals the quite surprising relationship between the property we have studied and the distributive law in the lattices of semigroup varieties. Namely, we have

Corollary 4.1. *Let $\mathcal{V}$ be a semigroup variety. If the sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups form a commutative semigroup with respect to the relational product, then either $\mathcal{V}$ consists of completely regular semigroups or the lattice $L(\mathcal{V})$ is distributive.*

Proof. According to the main theorem, we have to check that the lattice $L(\mathcal{V})$ is distributive whenever $\mathcal{V}$ satisfies one of the conditions 2)–6). It is well known (and easy to verify) that the lattice $L(\mathcal{P})$ has the following diagram:

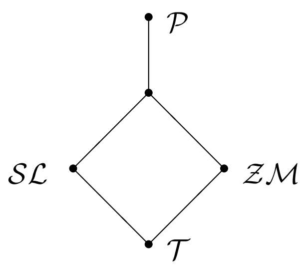

Figure 1. The lattice $L(\mathcal{P})$.

Thus, if $\mathcal{V}$ satisfies 2), that is, $\mathcal{V} = \mathcal{P}$ or $\mathcal{V} = \mathcal{P}^*$, then the lattice $L(\mathcal{V})$ is distributive. If $\mathcal{V}$ satisfies 3), then the distributivity of $L(\mathcal{V})$ is a consequence of a result due to Gerhard [2] who has proved that the lattice $L(\mathcal{SI})$ is distributive. If $\mathcal{V}$ satisfies one of the conditions 4)–6), then, as follows from [15, Proposition 1.15] and Propositions 3.5 and 3.6, the product of any two sub-semilattice fully invariant congruences on $\mathcal{V}$-free semigroups coincides with their set-theoretical union. Therefore the lattice of the sub-semilattice fully invariant congruences on the $\mathcal{V}$-free semigroup over the set X is distributive. This lattice is dually isomorphic to the interval $[\mathcal{SL} \wedge \mathcal{V}, \mathcal{V}]$ which is then distributive as well. As the variety $\mathcal{SL}$ is a neutral element of the lattice of semigroup varieties [7], the lattice $L(\mathcal{V})$ embeds into the direct product of $[\mathcal{SL} \wedge \mathcal{V}, \mathcal{V}]$ and the two-element lattice $L(\mathcal{SL})$. Hence $L(\mathcal{V})$ is distributive. □

Finally, we want to demonstrate how the technique employed in this paper may be used to give a simple proof for the description of the nil-semigroup varieties with distributive subvariety lattices found by the second author in [20].

Proposition 4.2. *A nil-variety $\mathcal{N}$ has distributive subvariety lattice if and only if $\mathcal{N}$ satisfies one of the identity systems* (0.8)–(0.28).

Proof. **Necessity.** By [14, Corollary 5], $\mathcal{N}$ is hereditarily homogeneous. In particular, for any positive integers m and n with $m \leq n$, the 0-transversal $W^0_{n,m}(\mathcal{N})$ is an $\mathbb{S}_m$-set. By Proposition 1.4, all these $\mathbb{S}_m$-sets are congruence distributive. Then, by Lemma 1.5, all non-empty transversals $W_{n,m}(\mathcal{N})$ are transitive.

Let us verify that $\mathcal{N}$ satisfies a non-trivial identity of the kind (0.30). Put $W = W_{3,3}(\mathcal{N})$ and $W^0 = W^0_{3,3}(\mathcal{N})$. We may assume that $W \neq \varnothing$: otherwise $\mathcal{N}$ would satisfy the identity $xyz = 0$, and therefore, all identities (0.30). Since W is an $\mathbb{S}_3$-subset of W^0, the lattice $\mathrm{Con}(W)$ is isomorphic to an interval of $\mathrm{Con}(W^0)$, whence $\mathrm{Con}(W)$ is distributive. By Corollary 1.7, the interval $[\mathrm{Perm}_3(\mathcal{N}), \mathbb{S}_3]$ of the lattice $\mathrm{Sub}(\mathbb{S}_3)$ is distributive too. Since the lattice $\mathrm{Sub}(\mathbb{S}_3)$ is non-distributive, the group $\mathrm{Perm}_3(\mathcal{N})$ must contain a non-trivial permutation.

Now we can complete the proof of necessity by repeating the arguments from the proofs of Lemmas 2.9, 2.11, 2.12 and Proposition 2.13. Indeed, these arguments only used the three following ingredients: the transitivity of non-empty transversals, the existence of a non-trivial permutation identity of length 3, and the fact that subvarieties of $\mathcal{N}$ are (n, m)-split for certain values of n and m, see Lemmas 2.8 and 2.10. We have observed that the first two properties hold true, and in place of the third one we have the stronger fact that $\mathcal{N}$ is hereditarily homogeneous.

Sufficiency immediately follows from Lemma 3.4 and Proposition 1.4. $\square$

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