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# SEMIGROUP VARIETIES ON WHOSE FREE OBJECTS ALMOST ALL FULLY INVARIANT CONGRUENCES ARE WEAKLY PERMUTABLE

**B. M. Vernikov***

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*A semigroup variety is said to be of index $\leqslant 2$ if all nil-semigroups of the variety are semigroups with zero multiplication. We describe all semigroup varieties $\mathcal{V}$ of index $\leqslant 2$ on free objects of which every two fully invariant congruences contained in the least semilattice congruence are weakly permutable, and semigroup varieties of index $\leqslant 2$ all of whose subvarieties share the above-mentioned property.*

In universal algebra, considerable attention is given to *congruence-permutable* and *weakly congruence-permutable* varieties, that is, varieties on all algebras of which every two congruences $\alpha$ and $\beta$ are, respectively, *permutable* (satisfy the equality $\alpha\beta = \beta\alpha$) and *weakly permutable* (satisfy the equality $\alpha\beta\alpha = \beta\alpha\beta$). The importance of these variety classes is principally determined by the fact that all varieties of groups and rings are congruence-permutable. However, as applied to varieties of yet another classical type of algebras — semigroup varieties — the conditions of being congruence-permutable and weakly congruence-permutable turn out to be too stringent, and essentially are not of interest. The point is that a semigroup variety is weakly congruence-permutable iff it consists of periodic groups. (This follows readily from relevant results of [1], and is a partial case of the results obtained in [2]; for congruence-permutable varieties, a similar fact was proven in [3].)

Nevertheless, the conditions of being congruence-permutable and weakly congruence-permutable can be weakened in a natural manner, by requiring that the corresponding equalities be satisfied not for all congruences on all semigroups in a variety but only for fully invariant congruences on semigroups free in the variety. We know from [4, 5] that the weaker conditions are satisfied by broad and important classes of semigroup varieties. Again, as stated in [4, 6] and a number of earlier publications dealing with identities in lattices of semigroup varieties, of importance are the varieties on free objects of which congruence-permutable or weakly congruence-permutable are not all fully invariant congruences, but only those that are contained in a least semilattice congruence (i.e., the least congruence the factor group with respect to which is a semilattice).

Semigroup varieties on whose free objects every two fully invariant congruences (contained in the least semilattice congruence) are permutable are said to be (*almost*) *$fi$-permutable*. Semigroup varieties on whose free objects every two fully invariant congruences (contained in the least semilattice congruence) are weakly permutable are said to be (*almost*) *weakly $fi$-permutable*. The conditions of being almost

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$fi$-permutable and almost weakly $fi$-permutable are, generally, not inherited by subvarieties (see, e.g., [7, Example 2.10]), as distinct from $fi$-permutability and weak $fi$-permutability the heredity of whom arises from trivial considerations (cf. [7, Lemma 1.1]). The varieties all of whose subvarieties are almost (weakly) $fi$-permutable are said to be *hereditarily almost* (*weakly*) *$fi$-permutable*. Notice that free objects of subvarieties in a variety $\mathcal{V}$ are exactly relatively free semigroups belonging to $\mathcal{V}$. Therefore the property of $\mathcal{V}$ being hereditarily almost (weakly) $fi$-permutable is equivalent to every two fully invariant congruences contained in the least semilattice congruence being (weakly) permutable on all relatively free semigroups of $\mathcal{V}$, and not only on $\mathcal{V}$-free semigroups. The description of $fi$-permutable and hereditarily almost $fi$-permutable varieties is given in [7], and of almost $fi$-permutable varieties — in [8].*

Recall that a semigroup variety is *completely regular* if it consists of *completely regular* semigroups (i.e., bands of groups). Similarly, a variety is said to be *completely simple* if it consists of completely simple semigroups, and is called a *nil-variety* if it consists of nil-semigroups. A *semigroup variety with completely regular square* is one the square of every semigroup of which is a completely regular semigroup. We say that a semigroup variety $\mathcal{V}$ has *index $n$* if all nil-semigroups of $\mathcal{V}$ are nilpotent of class at most $n$, where $n$ is least with this property. Clearly, completely regular varieties are exactly varieties with index 1.

A trivial argument will show that every almost weakly $fi$-permutable semigroup variety has a modular lattice of subvarieties (see Lemma 3 below). This fact and related results in [11-14] imply that every almost weakly $fi$-permutable variety either has index at most 2 or is close to nil-varieties. Weakly $fi$-permutable varieties of index at most 2 can be readily described based on results of [10]. In the present paper, we describe almost weakly $fi$-permutable and hereditarily almost weakly $fi$-permutable semigroup varieties with index not exceeding 2.

As usual, $\operatorname{var}\Sigma$ denotes a semigroup variety defined by a system $\Sigma$ of identities. Put

$$\mathcal{SL} = \operatorname{var}\{x^2 = x,\, xy = yx\}, \quad \mathcal{ZM} = \operatorname{var}\{xy = 0\},$$

$$\mathcal{P} = \operatorname{var}\{xy = x^2y,\, x^2y^2 = y^2x^2\}, \quad \overleftarrow{\mathcal{P}} = \operatorname{var}\{xy = xy^2,\, x^2y^2 = y^2x^2\}.$$

The symbols $\vee$ and $\wedge$ stand for, respectively, the union and the intersection in the lattice of all semigroup varieties, and in congruence lattices.

**THEOREM 1.** Let $\mathcal{V}$ be a semigroup variety of index at most 2. Then $\mathcal{V}$ is almost weakly $fi$-permutable if and only if one of the following conditions is met:

(1) $\mathcal{V}$ is a completely simple variety;

(2) $\mathcal{V}$ is a semigroup variety with completely regular square, containing $\mathcal{SL}$;

(3) $\mathcal{V} = \mathcal{A} \vee \mathcal{X}$, where $\mathcal{A}$ is a variety of periodic Abelian groups, and $\mathcal{X}$ is one of the varieties $\mathcal{P}$ or $\overleftarrow{\mathcal{P}}$;

(4) $\mathcal{V} = \mathcal{ZM}$.

**Proof.** Necessity. If $\mathcal{V} \not\supseteq \mathcal{SL}$ then the least semilattice congruence on each $\mathcal{V}$-free semigroup is the universal relation. Consequently, we have

**LEMMA 1.** If $\mathcal{V}$ is an almost (weakly) $fi$-permutable semigroup variety, and $\mathcal{V} \not\supseteq \mathcal{SL}$, then $\mathcal{V}$ is (weakly) $fi$-permutable.

**LEMMA 2** [10]. Each weakly $fi$-permutable semigroup variety either is a completely regular variety or is a nil-variety.

Let $\mathcal{V}$ be an arbitrary almost weakly $fi$-permutable semigroup variety with index not exceeding 2. Assume first that $\mathcal{V} \not\supseteq \mathcal{SL}$. By virtue of Lemma 1, $\mathcal{V}$ is weakly $fi$-permutable. By Lemma 2, $\mathcal{V}$ either is

*I have obtained a description of weakly $fi$-permutable varieties, but this result has not been brought to light to the full extent; its partial cases can be found in [9, 10].

completely regular or is a nil-variety. Clearly, in the latter case $\mathcal{V} = \mathcal{ZM}$, that is, condition (4) of Theorem 1 is met.

Let $\mathcal{V}$ be completely regular. In view of the basic result of [10], $\mathcal{V}$ either is a completely simple variety or is a variety of group semilattices. Each variety of group semilattices not containing $\mathcal{SL}$ is a variety of periodic groups; consequently, condition (1) of Theorem 1 holds in both of the two cases.

In proving the necessity of Theorem 1, we can assume that $\mathcal{V} \supseteq \mathcal{SL}$. As usual, $L(\mathcal{V})$ denotes the lattice of subvarieties of a variety $\mathcal{V}$.

**LEMMA 3.** If a semigroup variety $\mathcal{V}$ is almost $fi$-permutable then the lattice $L(\mathcal{V})$ is modular.

**Proof.** The variety $\mathcal{SL}$ is a neutral element of the lattice of all semigroup varieties. (This follows immediately, for instance, from [15].) Hence the lattice $L(\mathcal{V})$ embeds in the direct product of a 2-element lattice $L(\mathcal{SL})$ and an interval $[\mathcal{SL} \wedge \mathcal{V},\, \mathcal{V}]$ in $L(\mathcal{V})$. It suffices to verify that $[\mathcal{SL} \wedge \mathcal{V},\, \mathcal{V}]$ is modular. The interval is anti-isomorphic to the lattice of all fully invariant congruences on a $\mathcal{V}$-free semigroup of countable rank, contained in the least semilattice congruence on that semigroup. By virtue of Jónsson's results in [16], a lattice is modular if it has a presentation by weakly permutable equivalence relations (see also [17, Sec. IV.4]). Since the modular identity is self-dual, $[\mathcal{SL} \wedge \mathcal{V},\, \mathcal{V}]$ is modular.

Put $\mathcal{Q} = \mathrm{var}\,\{xy = x^2y,\, xyz^2 = yxz^2,\, xyx = yx^2\}$ and $\overleftarrow{\mathcal{Q}} = \mathrm{var}\,\{xy = xy^2,\, x^2yz = x^2zy,\, xyx = x^2y\}$. Relevant results of [13] (see also [14]) immediately imply the following:

**LEMMA 4.** If $\mathcal{V}$ is a semigroup variety of index at most 2, and the lattice $L(\mathcal{V})$ is modular, then one of the following is the case:

(1) $\mathcal{V}$ is a semigroup variety with completely regular square;

(2) $\mathcal{V} = \mathcal{D} \vee \mathcal{E}$, where $\mathcal{D}$ is one of the varieties $\mathcal{P}$ or $\mathcal{Q}$, and $\mathcal{E}$ is a completely regular variety;

(2′) $\mathcal{V} = \mathcal{D}' \vee \mathcal{E}'$, where $\mathcal{D}'$ is one of the varieties $\overleftarrow{\mathcal{P}}$ or $\overleftarrow{\mathcal{Q}}$, and $\mathcal{E}'$ is a completely regular variety.

Since $\mathcal{V} \supseteq \mathcal{SL}$, condition (2) of Theorem 1 holds in case (1). We verify that condition (3) of the same theorem is met in the above cases (2) and (2′). For symmetry reasons, it suffices to prove that in case (2) $\mathcal{V} = \mathcal{A} \vee \mathcal{P}$, where $\mathcal{A}$ is a variety of periodic Abelian groups.

Denote by $F$ an absolutely free semigroup of countable rank. Verification of the next lemma is straightforward.

**LEMMA 5.** Let $\mathcal{V}$ be a semigroup variety such that $\mathcal{V} \supseteq \mathcal{SL}$ and $\nu$, and let $\sigma$ be fully invariant congruences on $F$ complying with varieties $\mathcal{V}$ and $\mathcal{SL}$, respectively. The variety $\mathcal{V}$ is almost (weakly) $fi$-permutable if and only if every two fully invariant congruences on $F$, which contain $\nu$ and are contained in $\sigma$, are (weakly) permutable.

For every word $u \in F$, denote by $c(u)$ the set of all letters occurring in the representation of $u$, by $t(u)$ the terminal letter in the representation of $u$, by $\ell(u)$ the length of $u$, and by $\ell_x(u)$ ($\ell_i(u)$, resp.) the number of occurrences of the letter $x$ ($x_i$, resp.) in $u$. By $\equiv$ we denote the equality relation on $F$. For every natural $n > 1$, $\mathcal{A}_n$ is conceived of as a variety of all Abelian groups of exponent $n$. Below is a lemma needed for our further reasoning; its items (1)-(3) are well known and easily checked, and (4) was proven in [18].

**LEMMA 6.** The identity $u = v$ holds in the following varieties:

(1) $\mathcal{A}_n$ (iff $\ell_x(u) - \ell_x(v)$ is divisible by $n$ for every letter $x$);

(2) $\mathcal{SL}$ (iff $c(u) = c(v)$);

(3) $\mathcal{ZM}$ (iff either $u$ and $v$ are the same letter, or $\ell(u), \ell(v) \geqslant 2$);

(4) $\mathcal{P}$ (iff $c(u) = c(v)$, and either $\ell_{t(u)}(u), \ell_{t(v)}(v) > 1$, or $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ and $t(u) \equiv t(v)$).

Now let $\mathcal{V} = \mathcal{D} \vee \mathcal{E}$, where $\mathcal{D}$ is one of the varieties $\mathcal{P}$ or $\mathcal{Q}$, and $\mathcal{E}$ is a completely regular variety. It is well known that every variety $\mathcal{X}$ of periodic groups contains a greatest completely regular subvariety, which we denote by $\mathrm{CR}(\mathcal{X})$. There is no loss of generality in assuming that $\mathcal{E} = \mathrm{CR}(\mathcal{V})$. Moreover, $\mathcal{V} \supseteq \mathcal{D} \supseteq \mathcal{P} \supseteq \mathcal{SL}$, and so $\mathcal{E} \supseteq \mathcal{SL}$. Denote by $\rho$ and $\varepsilon$ the fully invariant congruences on $F$ complying with the varieties $\mathcal{P}$ and $\mathcal{E}$, respectively. In view o Lemma 5, $\rho$ and $\varepsilon$ are weakly permutable. Clearly, $\mathcal{P} \wedge \mathcal{E} = \mathrm{CR}(\mathcal{P}) = \mathcal{SL}$. By Lemma 6(2), $(u, v) \in \rho \vee \varepsilon$ iff $c(u) = c(v)$; in particular, $(xy, yx) \in \rho \vee \varepsilon = \rho\varepsilon\rho$. Hence $xy \mathrel{\rho} u \mathrel{\varepsilon} v \mathrel{\rho} yx$ for some words $u, v \in F$; in particular, $xy = u$ in $\mathcal{P}$. By virtue of Lemma 6(4), $c(u) = \{x, y\}$, $t(u) \equiv y$, $\ell_y(u) = 1$, and consequently, $u \equiv x^n y$, for some natural $n$. Similarly, $v \mathrel{\rho} yx$ implies $v \equiv y^m x$, for some natural $m$. Therefore $\mathcal{E}$ satisfies the identity

$$x^n y = y^m x, \tag{1}$$

and so $\mathcal{E}$ does not contain either the semigroup variety $\mathcal{LZ}$ of left zeros or the semigroup variety $\mathcal{RZ}$ of right zeros.

The next lemma, which is well known (see, e.g., [19, 20]), implies that $\mathcal{E} = \mathcal{A} \vee \mathcal{SL}$, where $\mathcal{A}$ is some variety of periodic groups.

**LEMMA 7.** If a completely regular semigroup variety does not contain the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ then either it is a group variety or a union of the group variety and the variety $\mathcal{SL}$.

Identity (1) holds in any group of $\mathcal{V}$. If, in this identity, we substitute 1 first for $x$ and then for $y$ we see that every group in $\mathcal{V}$ satisfies identities $y = y^m$ and $x^n = x$, and so therefore $xy = yx$. Consequently, $\mathcal{A}$ is a variety of periodic Abelian groups. In view of $\mathcal{RZ} \subseteq \mathcal{Q}$ and $\mathcal{RZ} \nsubseteq \mathcal{A} \vee \mathcal{SL} = \mathcal{E} = \mathrm{CR}(\mathcal{V})$, we conclude that $\mathcal{Q} \nsubseteq \mathcal{V}$. Hence $\mathcal{D} = \mathcal{P}$, and so $\mathcal{V} = \mathcal{D} \vee \mathcal{E} = \mathcal{P} \vee \mathcal{A} \vee \mathcal{SL} = \mathcal{A} \vee \mathcal{P}$.

Sufficiency. Independently in [4] and [5], it was proved that if a semigroup variety satisfies condition (1) of Theorem 1 then it is $fi$-permutable. In [6], it was stated that the varieties satisfying condition (2) of the same theorem are almost weakly $fi$-permutable. The case where condition (4) is met is evident: the lattice $L(\mathcal{ZM})$ is a 2-element chain, and so every two fully invariant congruences on each $\mathcal{ZM}$-free semigroup are comparable w.r.t. inclusion and, hence, permutable. It remains to consider the case where $\mathcal{V}$ satisfies condition (3).

By symmetry, we may assume that $\mathcal{V} = \mathcal{A} \vee \mathcal{P}$, where $\mathcal{A}$ is a variety of periodic Abelian groups.

**LEMMA 8** [7]. A variety $\mathcal{P}$ is hereditarily almost $fi$-permutable.

In view of Lemma 8, below we conceive of $\mathcal{A}$ as being non-trivial, that is, $\mathcal{A} = \mathcal{A}_n$ for some natural $n > 1$. We also need the following:

**LEMMA 9** [4, 5]. Every completely regular semigroup variety is almost $fi$-permutable.

Put $\mathcal{K} = \mathrm{CR}(\mathcal{A}_n \vee \mathcal{P})$. The variety $\mathcal{A}_n \vee \mathcal{P}$ satisfies the identities

$$x^{n+1} y = xy, \tag{2}$$

$$x^{n+1} y^{n+1} = y^{n+1} x^{n+1}. \tag{3}$$

Identity (3) shows that $\mathcal{K}$ does not contain any one of $\mathcal{LZ}$, $\mathcal{RZ}$. In virtue of Lemma 7, $\mathcal{K} = \mathcal{G} \vee \mathcal{SL}$, where $\mathcal{G}$ is the greatest group subvariety of $\mathcal{V}$. If we substitute 1 for $y$ in (2) we see that the exponent of $\mathcal{G}$ divides $n$. In view of (3), this implies that $\mathcal{G}$ is Abelian. Consequently, $\mathcal{G} \subseteq \mathcal{A}_n$. The converse is obvious, and so $\mathcal{G} = \mathcal{A}_n$. Thus $\mathcal{K} = \mathcal{A}_n \vee \mathcal{SL}$.

We know that $L(\mathcal{A}_n \vee \mathcal{SL}) \cong L(\mathcal{A}_n) \times L(\mathcal{SL})$ (see, e.g., [15]). It follows that every completely regular subvariety of $\mathcal{V}$ containing $\mathcal{SL}$ is of the form $\mathcal{G} \vee \mathcal{SL}$, where $\mathcal{G} \subseteq \mathcal{A}_n$. Denote by $\mathcal{T}$ the trivial variety. By [13, Lemma 15], each subvariety of $\mathcal{A}_n \vee \mathcal{P}$ has the form $\mathcal{X} \vee \mathcal{Y}$, where $\mathcal{X}$ is a completely regular variety, and $\mathcal{Y}$ is one of the varieties $\mathcal{T}$, $\mathcal{ZM}$, or $\mathcal{P}$. Thus each subvariety of $\mathcal{V}$ containing $\mathcal{SL}$ is a variety of one of the following six types: $\mathcal{SL}$, $\mathcal{G} \vee \mathcal{SL}$, $\mathcal{ZM} \vee \mathcal{SL}$, $\mathcal{P}$, $\mathcal{G} \vee \mathcal{ZM} \vee \mathcal{SL}$, $\mathcal{G} \vee \mathcal{P}$, where $\mathcal{G}$ is a non-trivial subvariety of $\mathcal{A}_n$.

Let $\alpha$ and $\beta$ be fully invariant congruences on a semigroup $F$ complying with some subvarieties of $\mathcal{V}$ containing $\mathcal{SL}$. By Lemma 5, it suffices to show that $\alpha$ and $\beta$ are weakly permutable. We may assume that $\alpha$ and $\beta$ are incomparable in the lattice of all fully invariant congruences on $F$. Denote by $\rho$, $\sigma$, and $\mu$ the fully invariant congruences on $F$ complying with $\mathcal{P}$, $\mathcal{SL}$, and $\mathcal{ZM}$, respectively. In what follows, $\gamma$, $\gamma_1$, and $\gamma_2$ will stand for fully invariant congruences on $F$ complying with non-trivial subvarieties of $\mathcal{A}_n$. The following nine cases are possible:

1. $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \sigma$;
2. $\alpha = \gamma \wedge \sigma$ and $\beta = \mu \wedge \sigma$;
3. $\alpha = \gamma \wedge \sigma$ and $\beta = \rho$;
4. $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \mu \wedge \sigma$;
5. $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \rho$;
6. $\alpha = \rho$ and $\beta = \gamma \wedge \mu \wedge \sigma$;
7. $\alpha = \gamma_1 \wedge \mu \wedge \sigma$ and $\beta = \gamma_2 \wedge \mu \wedge \sigma$;
8. $\alpha = \gamma_1 \wedge \mu \wedge \sigma$ and $\beta = \gamma_2 \wedge \rho$;
9. $\alpha = \gamma_1 \wedge \rho$ and $\beta = \gamma_2 \wedge \rho$.

Let $u, v \in F$ and $(u, v) \in \alpha \vee \beta$. We need only verify that $(u, v) \in \alpha\beta\alpha$ and $(u, v) \in \beta\alpha\beta$. Since $\alpha, \beta \subseteq \sigma$, in view of Lemma 6(2), $c(u) = c(v)$, and if $\ell(u) = \ell(v) = 1$, then $u \equiv v$. There is no loss of generality in assuming that $\ell(u) \geqslant 2$.

Case 1: $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \sigma$. The congruences $\alpha$ and $\beta$ are permutable by Lemmas 5 and 9.

Case 2: $\alpha = \gamma \wedge \sigma$ and $\beta = \mu \wedge \sigma$. If $\ell(v) \geqslant 2$ then $u \,\mu\, v$ by Lemma 6(3), and so $u \,\beta\, v$, that is, we may assume that $\ell(v) = 1$. Then $v \equiv x$ and $u \equiv x^k$ for some letter $x$ and some natural $k > 1$. Consequently, $u \,\beta\, x^{n+1} \,\alpha\, v$, that is, $(u, v) \in \beta\alpha$.

Case 3: $\alpha = \gamma \wedge \sigma$ and $\beta = \rho$. Let $x$ be an arbitrary letter occurring in the representation of $u$. By Lemma 6(4), $u \,\alpha\, ux^n \,\beta\, vx^n \,\alpha\, v$, that is, $(u, v) \in \alpha\beta\alpha$. It remains to show that $(u, v) \in \beta\alpha\beta$. Without loss of generality, we may assume that $c(u) = \{x_1, x_2, \ldots, x_m\}$. Let $t(u) \equiv x_i$ and $t(v) \equiv x_j$. If $\ell_i(u) > 1$ then $u \,\beta\, vx_i^n \,\alpha\, v$ by Lemma 6(4), that is, $(u, v) \in \beta\alpha$. Similarly we can verify that $(u, v) \in \alpha\beta$ if $\ell_j(v) > 1$. Lastly, if $\ell_i(u) = \ell_j(v) = 1$ then again we can appeal to Lemma 6(4) to arrive at

$$u \,\beta\, x_1 \cdots x_{i-1} x_{i+1} \cdots x_m x_i \,\alpha\, x_1 \cdots x_{j-1} x_{j+1} \cdots x_m x_j \,\beta\, v,$$

that is, $(u, v) \in \beta\alpha\beta$.

Case 4: $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \mu \wedge \sigma$. Put $\beta' = \gamma_2 \wedge \sigma$. Clearly, $(u, v) \in \alpha \vee \beta'$. In view of Lemmas 5 and 9, $\alpha$ and $\beta'$ are permutable, so $(u, v) \in \beta'\alpha$, that is, $u \,\beta'\, w \,\alpha\, v$ for some word $w \in F$. Obviously, $u \,\beta'\, w^{n+1} \,\alpha\, v$ and $\ell(w^{n+1}) > 1$. Lemma 6(3) implies that $u \,\mu\, w^{n+1}$, whence $u \,\beta\, w^{n+1} \,\alpha\, v$, that is, $(u, v) \in \beta\alpha$.

Case 5: $\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \rho$. Here, as well as in Case 9, we need the following:

**LEMMA 10.** Let $\gamma_1$ and $\gamma_2$ be fully invariant congruences on $F$ corresponding to some varieties of periodic Abelian groups, with $u, v \in F$, $(u, v) \in \gamma_1 \vee \gamma_2$, and $c(u) = c(v)$. Then there exist words $w_1, w_2 \in F$ such that $u \,\gamma_2\, w_1 \,\gamma_1\, w_2 \,\gamma_2\, v$, $u \,\rho\, w_1$, and $w_2 \,\rho\, v$.

**Proof.** Denote exponents of group varieties corresponding to congruences $\gamma_1$, $\gamma_2$, and $\gamma_1 \vee \gamma_2$ by $r$, $s$, and $t$, respectively. It is clear that $t$ is the greatest common divisor of the numbers $r$ and $s$. Without loss of generality, we may assume that $c(u) = c(v) = \{x_1, x_2, \ldots, x_m\}$. Let $i \in \{1, 2, \ldots, m\}$ and $\ell_i(u) < \ell_i(v)$. In virtue of Lemma 6(1), $\ell_i(v) - \ell_i(u) = g_i t$, for some natural $g_i$. Further, $r = kt$ and $s = \ell t$, for some coprime natural $k$ and $\ell$. Hence there exist natural numbers $a_i$ and $b_i$ such that $ka_i - \ell b_i = g_i$. Similarly we can verify the following: if $\ell_i(u) > \ell_i(v)$ then there exist natural $h_i$, $c_i$, and $d_i$ for which $\ell_i(u) - \ell_i(v) = h_i t$ and $kc_i - \ell d_i = h_i$.

Put $u_0 \equiv u$ and $v_0 \equiv v$. For every $i = 1, 2, \ldots, m$, we then use induction to define the words

$$u_i \equiv \begin{cases} u_{i-1} & \text{if } \ell_i(u) \leqslant \ell_i(v); \\ x_i^{d_i s} u_{i-1} & \text{if } \ell_i(u) > \ell_i(v), \end{cases} \qquad v_i \equiv \begin{cases} v_{i-1} & \text{if } \ell_i(v) \leqslant \ell_i(u); \\ x_i^{b_i s} v_{i-1} & \text{if } \ell_i(v) > \ell_i(u). \end{cases}$$

Lastly, put $w_1 \equiv u_m$ and $w_2 \equiv v_m$. Obviously, $\ell_i(w_1) - \ell_i(u)$ and $\ell_i(w_2) - \ell_i(v)$ are divisible by $s$ for every $i \in \{1, 2, \ldots, m\}$. In view of Lemma 6(1), this means that $u \, \gamma_2 \, w_1$ and $w_2 \, \gamma_2 \, v$. We verify $w_1 \, \gamma_1 \, w_2$. By Lemma 6(1), we need to state that $\ell_i(w_1) - \ell_i(w_2)$ is divisible by $r$ for all $i \in \{1, 2, \ldots, m\}$. Indeed, let $i \in \{1, 2, \ldots, m\}$. If $\ell_i(u) = \ell_i(v)$ then $\ell_i(w_1) = \ell_i(u) = \ell_i(v) = \ell_i(w_2)$ and $\ell_i(w_1) - \ell_i(w_2) = 0$. For $\ell_i(u) > \ell_i(v)$, we have $\ell_i(w_1) = \ell_i(u) + d_i s$ and $\ell_i(w_2) = \ell_i(v)$. Therefore

$$\ell_i(w_1) - \ell_i(w_2) = \ell_i(u) - \ell_i(v) + d_i s = h_i t + d_i s = c_i kt - d_i \ell t + d_i \ell t = c_i kt = c_i r.$$

Finally, if $\ell_i(u) < \ell_i(v)$ then $\ell_i(w_1) = \ell_i(u)$ and $\ell_i(w_2) = \ell_i(v) + b_i s$, whence

$$\ell_i(w_1) - \ell_i(w_2) = \ell_i(u) - \ell_i(v) - b_i s = -g_i t - b_i s = -a_i kt + b_i \ell t - b_i \ell t = -a_i kt = -a_i r.$$

Thus $u \, \gamma_2 \, w_1 \, \gamma_1 \, w_2 \, \gamma_2 \, v$.

We now check that $u \, \rho \, w_1$. By the construction of $w_1$, $c(u) = c(w_1)$ and $t(u) \equiv t(w_1)$. Let $t(u) \equiv x_i$. If $\ell_i(u) = 1$ then $\ell_i(u) \leqslant \ell_i(v)$ and $\ell_i(w_1) = \ell_i(u) = 1$. And if $\ell_i(u) > 1$ then $\ell_i(w_1) \geqslant \ell_i(u) > 1$. By Lemma 6(4), it follows that $u \, \rho \, w_1$. A check on $w_2 \, \rho \, v$ is similar.

Put $\beta' = \gamma_2 \wedge \sigma$. It is clear that $(u, v) \in \alpha \vee \beta'$. In view of Lemmas 5 and 9, $\alpha$ permutes with $\beta'$, so $(u, v) \in \alpha\beta'$, that is, $u \, \alpha \, w \, \beta' \, v$ for some word $w \in F$. Let $x \in c(u)$; then $wx^n \, \beta' \, vx^n$, and in view of Lemma 6(4), $wx^n \, \rho \, vx^n$. It follows that $wx^n \, \beta \, vx^n$, hence $u \, \alpha \, ux^n \, \alpha \, wx^n \, \beta \, vx^n \, \alpha \, v$, that is, $(u, v) \in \alpha\beta\alpha$.

We are left to verify that $(u, v) \in \beta\alpha\beta$. Let $t(u) \equiv x_i$ and $t(v) \equiv x_j$. If $\ell_j(v) > 1$ then $wx^n \, \rho \, v$ by Lemma 6(4). In virtue of $u \, \alpha \, wx^n \, \beta' \, v$, we have $u \, \alpha \, wx^n \, \beta \, v$, that is, $(u, v) \in \alpha\beta$. Similarly we can verify that $\ell_i(u) > 1$ implies $(u, v) \in \beta\alpha$. Therefore we may assume that $\ell_i(u) = \ell_j(v) = 1$. Let $w_1$ and $w_2$ be as in Lemma 10. Then $u \, \gamma_2 \, w_1 \, \gamma_1 \, w_2 \, \gamma_2 \, v$, $u \, \rho \, w_1$, and $w_2 \, \rho \, v$. The last two relations imply, in particular, that $c(w_1) = c(u) = c(v) = c(w_2)$. Hence $w_1 \, \sigma \, w_2$ and $w_1 \, \alpha \, w_2$. Moreover, $u \, \beta \, w_1$ and $w_2 \, \beta \, v$. Consequently, $(u, v) \in \beta\alpha\beta$.

Case 6: $\alpha = \rho$ and $\beta = \gamma \wedge \mu \wedge \sigma$. Clearly, $\alpha \vee \beta \subseteq \mu$, and so $u \, \mu \, v$. In view of Lemma 6(3), $\ell(v) \geqslant 2$. Put $\beta' = \gamma \wedge \sigma$. Obviously, $(u, v) \in \alpha \vee \beta'$. By virtue of Case 3, $\alpha$ weakly permutes with $\beta'$. It follows that $(u, v) \in \alpha\beta'\alpha$, that is, $u \, \alpha \, w_1 \, \beta' \, w_2 \, \alpha \, v$ for some words $w_1, w_2 \in F$. We have $\alpha = \rho \subseteq \mu$, yielding $u \, \mu \, w_1$. Since $\ell(u) > 1$, in view of Lemma 6(3), it follows that $\ell(w_1) > 1$. Likewise we can verify that $\ell(w_2) > 1$. Consequently, $w_1 \, \mu \, w_2$, so $u \, \alpha \, w_1 \, \beta \, w_2 \, \alpha \, v$, that is, $(u, v) \in \alpha\beta\alpha$. Further, $(u, v) \in \beta'\alpha\beta'$, that is, $u \, \beta' \, w_1' \, \alpha \, w_2' \, \beta' \, v$ for some words $w_1', w_2' \in F$. Let $x \in c(u)$ as above. Then $u \, \beta' \, w_1'x^n \, \alpha \, w_2'x^n \, \beta' \, v$, $u \, \mu \, w_1'x^n$, and $w_2'x^n \, \mu \, v$, whence $u \, \beta \, w_1'x^n \, \alpha \, w_2'x^n \, \beta \, v$, that is, $(u, v) \in \beta\alpha\beta$.

Case 7: $\alpha = \gamma_1 \wedge \mu \wedge \sigma$ and $\beta = \gamma_2 \wedge \mu \wedge \sigma$. As in Case 6, $u \, \mu \, v$, and by Lemma 6(3), $\ell(v) \geqslant 2$. Put $\alpha' = \gamma_1 \wedge \sigma$ and $\beta' = \gamma_2 \wedge \sigma$. In view of Lemmas 5 and 9, $\alpha'$ permutes with $\beta'$, whence $(u, v) \in \alpha'\beta'$, that is, $u \, \alpha' \, w \, \beta' \, v$ for some word $w \in F$. Let $x \in c(u)$; then $u \, \alpha' \, wx^n \, \beta' \, v$ and $u \, \mu \, wx^n \, \mu \, v$, and so $u \, \alpha \, wx^n \, \beta \, v$. Consequently, $(u, v) \in \alpha\beta$.

Case 8: $\alpha = \gamma_1 \wedge \mu \wedge \sigma$ and $\beta = \gamma_2 \wedge \rho$. As in the previous two cases, $u \, \mu \, v$, and in view of Lemma 6(3), $\ell(v) \geqslant 2$. Put $\alpha' = \gamma_1 \wedge \sigma$. It is clear that $(u, v) \in \alpha' \vee \beta$. By the argument in Case 5, $\alpha'$ weakly permutes with $\beta$. It follows that $(u,v) \in \alpha'\beta\alpha'$, that is, $u\,\alpha'\,w_1\,\beta\,w_2\,\alpha'\,v$ for some words $w_1, w_2 \in F$. Let $x \in c(u)$; then $u\,\alpha'\,w_1 x^n\,\beta\,w_2 x^n\,\alpha'\,v$, $u\,\mu\,w_1 x^n$, and $w_2 x^n\,\mu\,v$, whence $u\,\alpha\,w_1 x^n\,\beta\,w_2 x^n\,\alpha\,v$. Thus $(u,v) \in \alpha\beta\alpha$. Further, $(u,v) \in \beta\alpha'\beta$, that is, $u\,\beta\,w_1'\,\alpha'\,w_2'\,\beta\,v$ for some words $w_1', w_2' \in F$. Since $\beta \subseteq \rho \subseteq \mu$, we obtain $u\,\mu\,w_1'$. In virtue of $\ell(u) > 1$ and Lemma 6(3), $\ell(w_1') > 1$. Likewise we can verify that $\ell(w_2') > 1$. Consequently, $w_1'\,\mu\,w_2'$, so $u\,\beta\,w_1'\,\alpha\,w_2'\,\beta\,v$, that is, $(u,v) \in \beta\alpha\beta$.

Case 9: $\alpha = \gamma_1 \wedge \rho$ and $\beta = \gamma_2 \wedge \rho$. By symmetry, it suffices to establish that $(u,v) \in \beta\alpha\beta$. Clearly, $u\,\rho\,v$ in this instance. Let $w_1$ and $w_2$ be as in Lemma 10. Then $u\,\gamma_2\,w_1\,\gamma_1\,w_2\,\gamma_2\,v$, $u\,\rho\,w_1$, and $w_2\,\rho\,v$. The last two relations imply that $w_1\,\rho\,u\,\rho\,v\,\rho\,w_2$, that is, $w_1\,\rho\,w_2$. Consequently, $u\,\beta\,w_1\,\alpha\,w_2\,\beta\,v$ and $(u,v) \in \beta\alpha\beta$.

**THEOREM 2.** Let $\mathcal{V}$ be a semigroup variety of index at most 2. Then the following conditions are equivalent:

(1) $\mathcal{V}$ is hereditarily almost weakly $fi$-permutable;

(2) $\mathcal{V}$ is hereditarily almost $fi$-permutable;

(3) $\mathcal{V}$ either is a completely regular variety or is contained in one of $\mathcal{P}$, $\overleftarrow{\mathcal{P}}$.

**Proof.** (3) $\Rightarrow$ (2) follows from Lemmas 8 and 9.

(2) $\Rightarrow$ (1) is evident.

To verify (1) $\Rightarrow$ (3), we need the following strengthening of Lemma 2.3 in [7].

**LEMMA 11.** If $\mathcal{V}$ is a hereditarily almost weakly $fi$-permutable semigroup variety, then either $\mathcal{V}$ is completely regular, or $\mathcal{V}$ does not contain any non-trivial completely simple subvarieties.

**Proof.** Let $\mathcal{V}$ be a hereditarily almost weakly $fi$-permutable semigroup variety that is not completely regular, that is, $\mathcal{V} \supseteq \mathcal{ZM}$. Let $\mathcal{X}$ be an arbitrary completely simple subvariety of $\mathcal{V}$. Then $\mathcal{X} \vee \mathcal{ZM} \subseteq \mathcal{V}$, and hence the variety $\mathcal{X} \vee \mathcal{ZM}$ is almost weakly $fi$-permutable. Moreover, $\mathcal{X} \vee \mathcal{ZM} \not\supseteq \mathcal{SL}$. By Lemma 1, $\mathcal{X} \vee \mathcal{ZM}$ is weakly $fi$-permutable, and is a nil-variety by Lemma 2. Consequently, the variety $\mathcal{X}$ is trivial.

Let $\mathcal{V}$ be a hereditarily almost weakly $fi$-permutable semigroup variety. In view of Lemma 3, the lattice $L(\mathcal{V})$ is modular, and so one of the cases (1), (2), or (2$'$) specified in Lemma 4 will hold for $\mathcal{V}$. From [7, proof of Lemma 2.5], it follows that every semigroup variety with completely regular square, which is not completely regular and does not contain any non-trivial completely simple subvarieties, is contained in $\mathcal{SL} \vee \mathcal{ZM}$. This, combined with Lemma 11, implies the following: if Lemma 4(1) holds then either $\mathcal{V}$ is completely regular, or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{ZM} \subseteq \mathcal{P}$. By symmetry, it remains to consider the situation where Lemma 4(2) holds for $\mathcal{V}$, that is, $\mathcal{V} = \mathcal{D} \vee \mathcal{E}$, where $\mathcal{D}$ is one of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{E}$ is completely regular. Lemma 11 allows us to assume that $\mathcal{V}$ is freed of non-trivial completely simple subvarieties; so, $\mathcal{E} \subseteq \mathcal{SL}$. For the same reason, $\mathcal{RZ} \nsubseteq \mathcal{V}$. Since $\mathcal{RZ} \subseteq \mathcal{Q}$, we obtain $\mathcal{D} \neq \mathcal{Q}$ and $\mathcal{D} = \mathcal{P}$. Moreover, $\mathcal{E} \subseteq \mathcal{SL} \subseteq \mathcal{P} = \mathcal{D}$; hence, $\mathcal{V} = \mathcal{D} \vee \mathcal{E} = \mathcal{D} = \mathcal{P}$. The theorem is proved.

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