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A WEAKER VERSION OF CONGRUENCE-PERMUTABILITY FOR SEMIGROUP VARIETIES

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Congruences α and β are 2.5*-permutable if $\alpha \vee \beta = \alpha\beta \cup \beta\alpha$, where $\vee$ is a union in the congruence lattice and $\cup$ is the set-theoretic union. A semigroup variety $\mathcal{V}$ is fi-permutable (fi-*2.5*-permutable) if every two fully invariant congruences are permutable (*2.5*-permutable) on all $\mathcal{V}$-free semigroups. Previously, a description has been furnished for fi-permutable semigroup varieties. Here, it is proved that a semigroup variety is fi-*2.5*-permutable iff it either consists of completely simple semigroups, or coincides with a variety of all semilattices, or is contained in one of the explicitly specified nil-semigroup varieties. As a consequence we see that (a) for semigroup varieties that are not nil-varieties, the property of being fi-*2.5*-permutable is equivalent to being fi-permutable; (b) for a nil-variety $\mathcal{V}$, if the lattice $L(\mathcal{V})$ of its subvarieties is distributive then $\mathcal{V}$ is fi-*2.5*-permutable; (c) if $\mathcal{V}$ is combinatorial or is not completely simple then the fact that $\mathcal{V}$ is fi-*2.5*-permutable implies that $L(\mathcal{V})$ belongs to a variety generated by a* 5*-element modular non-distributive lattice.*

It is well known that satisfaction of identities in lattices of varieties of universal algebras is closely connected with multiplicative properties of fully invariant congruences on free algebras of the varieties. (The case in point are the properties expressible in terms of products of binary relations.) Let α and β be congruences on a same algebra and n be a natural number. Put $\alpha \circ_n \beta = \alpha\beta\alpha\beta\cdots$, where the number of factors on the right-hand side of the equality is equal to n. The congruences α and β are said to be *n-permutable* if $\alpha \circ_n \beta = \beta \circ_n \alpha$. For $n = 2$, ordinary permutability obtains; 3-permutable congruences are conventionally said to be *weakly permutable*. The classical results by Jónsson maintain that if a variety $\mathcal{V}$ is (*weakly*) *congruence-permutable*, that is, if every two congruences are (weakly) permutable on any algebra in $\mathcal{V}$, then the lattice $L(\mathcal{V})$ of subvarieties of $\mathcal{V}$ is Arguesian (modular); see, e.g., [1, Ch. IV, Sec. 4]. In [2] it was shown that the *congruence-n-permutability* of $\mathcal{V}$, that is, n-permutability of every two congruences on any algebra in $\mathcal{V}$, implies the existence of a non-trivial identity in $L(\mathcal{V})$ (for any n).

For the case of semigroup varieties, however, the multiplicative restrictions imposed on all congruences of all semigroups in a variety generally appear to be too stringent and are of no interest from the standpoint of semigroup theory. Specifically, a semigroup variety is congruence-n-permutable iff it is a variety of periodic groups. (For $n = 2$, this was proved in [3], and for an arbitrary n — in [2].)

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The situation becomes more interesting if multiplicative restrictions are imposed not on all but on fully invariant congruences only, and on free objects rather than on any semigroups of a given variety. In this event, on the one hand, the connections with identities in the lattices of varieties are preserved to the full extent, and on the other there arise extensive and important classes of varieties. Results concerning multiplicative properties of fully invariant congruences on relatively free semigroups have been applied in studying identities in the lattices of semigroup varieties (see, e.g., [4-6]).

A general problem arises naturally calling for studying semigroup varieties the fully invariant congruences on the free objects of which satisfy one or another multiplicative restriction. A number of results in this direction were obtained in [7-9]. In particular, [8] contains a complete description of semigroup varieties with permutable fully invariant congruences on their free objects. Some of the new results concerning this subject have been announced in [10], and we devote the present article to discussing one of them.

We need some notation and definitions. Let α and β be congruences on a same algebra. As is known, their union $\alpha \vee \beta$ in the congruence lattice is expressed, via relations of the form $\alpha \circ_n \beta$, as follows:

$$\alpha \vee \beta = \alpha \cup \beta \cup \alpha\beta \cup \beta\alpha \cup \alpha\beta\alpha \cup \beta\alpha\beta \cup \cdots \cup \alpha \circ_n \beta \cup \beta \circ_n \alpha \cup \alpha \circ_{n+1} \beta \cup \cdots , \qquad (0.1)$$

where $\cup$ is the set-theoretic union. Clearly, if α and β are n-permutable then $\alpha \vee \beta = \alpha \circ_n \beta$. From the standpoint of (0.1), it seems natural to consider the following property:

$$\alpha \vee \beta = \alpha \circ_n \beta \cup \beta \circ_n \alpha \qquad (0.2)$$

(equivalent to $\alpha \circ_{n+1} \beta = \alpha \circ_n \beta \cup \beta \circ_n \alpha$). The property in question is weaker than n-permutability and is stronger than $(n+1)$-permutability. Equality (0.1) shows that this is probably the sole natural restriction on congruences lying "in-between" the n-permutability and the $(n+1)$-permutability. Therefore congruences α and β with property (0.2) will be referred to as $n.5$-permutable. Specifically, α and β are 2.5-permutable, if $\alpha \vee \beta = \alpha\beta \cup \beta\alpha$, and are 1.5-permutable if $\alpha \vee \beta = \alpha \cup \beta$.

As usual, $\mathbb{N}$ denotes the set of all natural numbers. We put $\overline{\mathbb{N}} = \mathbb{N} \cup \{n + 0.5 \mid n \in \mathbb{N}\}$. A semigroup variety every two fully invariant congruences on all free objects of which are r-permutable (with $r \in \overline{\mathbb{N}}$) are said to be fi-r-permutable. If $r = 2$ (resp., $r = 3$) then fi-r-permutable varieties are referred to as (*weakly*) *fi-permutable.*

In the present article we provide a complete description for fi-2.5-permutable semigroup varieties. Notice that the condition of being 2.5-permutable for fully invariant congruences on free objects of a variety has already appeared in a number of works (see, e.g., [7, 11]). A semigroup variety is said to be *completely simple* if it consists of completely simple semigroups. We write $\mathcal{SL}$ for the variety of all semilattices. Our basic result is the following:

THEOREM. A semigroup variety $\mathcal{V}$ is fi-2.5-permutable if and only if either $\mathcal{V}$ is a completely simple variety, or $\mathcal{V} = \mathcal{SL}$, or $\mathcal{V}$ satisfies one of the following systems of identities:

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = 0,$ (0.3)

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ xy^2 = 0,$ (0.4)

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = xy^2,\ x^2yz = 0,$ (0.5)

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = yx^2,\ x^3y = 0,$ (0.6)

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = yx^2,\ x^2y^2 = 0,$ (0.7)

$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = yx^2,\ x^3y = x^2y^2,\ x^2y^2z = 0,$ (0.8)

$xyz = zyx,\ xyx = 0,$ (0.9)

$xyz = zyx,\ x^2y = yxy,\ x^2yz = 0,$ (0.10)

$$xyz = zyx,\ x^2y = xyx,\ x^3y = 0, \tag{0.11}$$
$$xyz = zyx,\ x^2y = xyx,\ x^2y^2 = 0, \tag{0.12}$$
$$xyz = zyx,\ x^2y = xyx,\ x^3y = x^2y^2,\ x^2y^2z = 0, \tag{0.13}$$
$$xyz = yzx,\ x^3y = 0, \tag{0.14}$$
$$xyz = yzx,\ x^2y^2 = 0, \tag{0.15}$$
$$xyz = yzx,\ x^3y = x^2y^2,\ x^2y^2z = 0, \tag{0.16}$$
$$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = y^2x,\ \ xyzt = 0, \tag{0.17}$$
$$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ xy^2 = yx^2,\ \ xyzt = 0, \tag{0.18}$$
$$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi},\ x^2y = yx^2,\ \ x_1x_2x_3x_4x_5 = 0, \tag{0.19}$$
$$xyz = zyx,\ \ xyx = yxy,\ \ xyzt = 0, \tag{0.20}$$
$$xyz = zyx,\ \ x^2y = xyx,\ \ x_1x_2x_3x_4x_5 = 0, \tag{0.21}$$
$$xyz = yzx,\ \ x_1x_2x_3x_4x_5 = 0, \tag{0.22}$$

where in systems (0.3) and (0.17), π is one of the permutations (12), (13), or (23), and in systems (0.4)-(0.8), (0.18), and (0.19), π is one of (12), (23).

It is worth mentioning that some auxiliary results of this article pertain to weakly fi-permutable varieties, or even to fi-r-permutable varieties for any $r \in \overline{\mathbb{N}}$.

The varieties specified by systems (0.3)-(0.22) are *nil-varieties*, that is, they consist of nil-semigroups. We will see later that exactly this case is most complicated taking up a major share of the proof. Placed at the center stage here are some results from [12-14]. In [12, 13], it was shown that the structure of lattices of nil-varieties is largely determined by a structure of congruence lattices of some unary algebras of a special sort, the so-called G-sets. Lattice and multiplicative properties of congruences on G-sets were dealt with in [14].

In Sec. 1, we reproduce some relevant results from [12-14], and prove multiplicative analogs for those in [12, 13]. In Secs. 2 and 3, we present the proof of the theorem, and give a number of its corollaries in Sec. 4.

1. PRELIMINARIES

1.1. Congruences on G-sets. Let A be a non-empty set, G a group, and φ a homomorphism from G to a group of all permutations on A. With every element $g \in G$ we associate a unary operation g^* on A given by the rule $g^*(a) = (\varphi(g))(a)$, for every $a \in A$. A unary algebra with support A and set $\{g^* \mid g \in G\}$ of operations is called a *G-set*. A congruence lattice of a G-set A is denoted by $\mathrm{Con}(A)$. A G-set A is said to be *transitive* if, for any $x, y \in A$, there exists an element $g \in G$ such that $y = g^*(x)$. A transitive G-subset of a G-set A is called an *orbit* in A.

A G-set A is *segregated* if, for any congruence α on A and for two distinct orbits B and C in A, the following condition holds: if $b\,\alpha\,c$ for some elements $b \in B$ and $c \in C$ then $x\,\alpha\,y$ for any elements $x, y \in B \cup C$. Proposition 1.3 in [14] yields

LEMMA 1.1. If a G-set is segregated then its every two distinct non one-element orbits are not isomorphic.

Obviously, the following holds:

LEMMA 1.2. If a G-set A contains at most one non one-element orbit then it is segregated.

A G-set A is *congruence-r-permutable* (with $r \in \overline{\mathbb{N}}$) if every two congruences on A are r-permutable. If $r = 2$ (resp., $r = 3$) then congruence-r-permutable G-sets are, as usual, said to be (*weakly*) *congruence-permutable*. Denote by $\mathbf{M}_3$ the variety of lattices generated by a 5-element modular non-distributive lattice. The propositions below are partial cases of Theorems 2.2 and 3.4 in [14], respectively.

Proposition 1.1. The congruence lattice of a G-set A belongs to $\mathbf{M}_3$ if and only if A is segregated and contains at most three orbits, and a congruence lattice of each orbit of this G-set belongs to $\mathbf{M}_3$.

Proposition 1.2. Let $r \in \{2.5, 3\}$. A G-set A is congruence-r-permutable if and only if it is segregated and contains at most three orbits, and each orbit of that G-set is congruence-r-permutable.

Propositions 1.1 and 1.2 describe G-sets with the properties treated, in these sets, w.r.t. orbits, that is, transitive G-subsets of a given G-set. In the remaining part of this subsection we deal with congruences of the transitive G-sets. Let A be a G-set and $a \in A$. Put $\mathrm{Stab}_A(a) = \{g \in G \mid g^*(a) = a\}$. Clearly, $\mathrm{Stab}_A(a)$ is a subgroup of G. As usual, $\mathrm{Sub}(G)$ denotes the lattice of subgroups of a group G. We need the following well-known statement (see, e.g., [15, Lemma 4.20]).

LEMMA 1.3. The congruence lattice of a transitive G-set A is isomorphic to an interval $[\mathrm{Stab}_A(a),\, G]$ of the lattice $\mathrm{Sub}(G)$, where a is an arbitrary element of A.

If P and Q are subsets of a group G then we put $PQ = \{pq \mid p \in P,\, q \in Q\}$. Let H_1 and H_2 be subgroups of G and n be a natural number. Assume $H_1 \circ_n H_2 = H_1H_2H_1H_2\cdots$, where the number of factors on the right-hand side of the equality is equal to n. The subgroups H_1 and H_2 are said to be *n-permutable*, if $H_1 \circ_n H_2 = H_2 \circ_n H_1$, and n.5-permutable if $H_1 \vee H_2 = H_1 \circ_n H_2 \cup H_2 \circ_n H_1$, where $\vee$ is a union in $\mathrm{Sub}(G)$ and $\cup$ is the set-theoretic union. Lemma 2.10 in [7] immediately implies the following multiplicative analog for Lemma 1.3.

LEMMA 1.4. Let $r \in \overline{\mathbb{N}}$. A transitive G-set A is congruence-r-permutable if and only if any two groups in the interval $[\mathrm{Stab}_A(a),\, G]$ of $\mathrm{Sub}(G)$ are r-permutable, where a is an arbitrary element of A.

As usual, $\mathbf{S}_n$ denotes a symmetric group of degree n. Verification of the next lemma is straightforward.

LEMMA 1.5. Subgroups of the group $\mathbf{S}_3$ generated by its two distinct transpositions are not 2.5-permutable.

We also need the following:

LEMMA 1.6. If a transitive G-set A contains at most three elements then its congruence lattice contains at most two.

Proof. If $|A| \leqslant 2$ then the conclusion is obvious. Let $A = \{x, y, z\}$. Consider a congruence α on A that is not an equality relation. There is no loss of generality in assuming that $x \,\alpha\, y$. Since A is transitive, there exists a $g \in G$ such that $z = g^*(x)$. The case where $g^*(y) = z$ is impossible, for in this instance $g^*(x) = g^*(y)$, and so $x = y$. Consequently, $z = g^*(x) \,\alpha\, g^*(y) \in \{x, y\}$. Hence α is a universal relation.

1.2. Lattices of semigroup nil-varieties. In this subsection we deal with semigroups with zero in the signature. However, all that we say in the discussion that follows will also be valid for ordinary semigroup varieties since, as shown in [16], the lattice of semigroup nil-varieties with zero in the signature is isomorphic to a lattice of semigroup nil-varieties in the usual semigroup signature.

Hereinafter, F denotes a free semigroup over the alphabet $\{x_1, x_2, \ldots, x_m, \ldots\}$. The equality in F is denoted $\equiv$. We need the following notation related to an arbitrary non-zero word u: $\ell(u)$ is the length of u; $\ell_x(u)$ is the number of occurrences of the letter x in u; $c(u)$ is the set of all letters occurring in the representation of u; $n(u)$ is the number of letters occurring in the representation of u. We also say the following: a word u *divides* a word v (written $u \lhd v$) if $v \equiv a\xi(u)b$ for some (possibly empty) words a and b and some endomorphism ξ of a semigroup F; a word u *divides a word v in a variety* $\mathcal{V}$ (written $u \overset{\mathcal{V}}{\lhd} v$) if u divides some word w such that $v = w$ in $\mathcal{V}$; words u and v are *similar* (written $u \approx v$) if one is obtained from the other by renaming letters; words u and v are *similar in* $\mathcal{V}$ (written $u \overset{\mathcal{V}}{\approx} v$) if there exists a word w such that $u \approx w$ and $v = w$ in $\mathcal{V}$.

Let m and n be natural numbers and $m \leqslant n$. A semigroup variety $\mathcal{V}$ is said to be (n,m)-*splittable* if the fact that an identity $u = v$ such that $\ell(u) = n$, $n(u) = m$, and $\ell(v) > n$ holds in $\mathcal{V}$ implies that $u = 0$ also holds in $\mathcal{V}$. A variety that is (n,m)-splittable for all n and m such that $n \geqslant m$ is said to be *homogeneous*; a variety all of whose subvarieties are (n,m)-splittable (homogeneous) is *hereditarily* (n,m)-*splittable* (*hereditarily homogeneous*). It is easy to see that every hereditarily homogeneous variety is a nil-variety.

For brevity, we write '$u \neq 0$ in $\mathcal{V}$' to signify the fact that $u = 0$ does not hold in $\mathcal{V}$. Put

$$F_{n,m}(\mathcal{V}) = \{u \in F \mid \ell(u) = n,\ c(u) = \{x_1, x_2, \ldots, x_m\} \text{ and } u \neq 0 \text{ in } \mathcal{V}\}.$$

Let $W_{n,m}(\mathcal{V})$ be a subset in $F_{n,m}(\mathcal{V})$ with the following property: for every $u \in F_{n,m}(\mathcal{V})$, there exists only one word u^* such that $u = u^*$ holds in $\mathcal{V}$. Put

$$W^0_{n,m}(\mathcal{V}) = W_{n,m}(\mathcal{V}) \cup \{0\}.$$

A set like $W_{n,m}(\mathcal{V})$ is called a *transversal*, and a set like $W^0_{n,m}(\mathcal{V})$ — a 0-*transversal*. Notice that the set $F_{n,m}(\mathcal{V})$ may be empty (if all words of length n in m letters are equal to 0 in $\mathcal{V}$). In this instance the set $W_{n,m}(\mathcal{V})$ is also empty, but $W^0_{n,m}(\mathcal{V})$ is always non-empty for it contains 0.

We define the action of a group $\mathbf{S}_m$ on $W^0_{n,m}(\mathcal{V})$. If $u \in F$, $c(u) = \{x_1, x_2, \ldots, x_m\}$, and $\sigma \in \mathbf{S}_m$, then $u\sigma$ denotes the image of a word u under the endomorphism on a semigroup F extending the map $x_i \longmapsto x_{i\sigma}$ (we assume that $i\sigma = i$ for $i > m$). If $u \in F_{n,m}(\mathcal{V})$ then $u\sigma \in F_{n,m}(\mathcal{V})$, and we can consider a word $(u\sigma)^*$. For every $\sigma \in \mathbf{S}_m$, put $\sigma^*(u) \equiv (u\sigma)^*$, where $u \in W_{n,m}(\mathcal{V})$, and $\sigma^*(0) \equiv 0$. It is easy to check that $W^0_{n,m}(\mathcal{V})$ with the set $\{\sigma^* \mid \sigma \in \mathbf{S}_m\}$ of operations is an $\mathbf{S}_m$-set if $\mathcal{V}$ is (n,m)-splittable (which follows from [12, proof of Lemma 1.1]). If, in addition, $W_{n,m}(\mathcal{V}) \neq \varnothing$ then $W_{n,m}(\mathcal{V})$ is an $\mathbf{S}_m$-subset of $W^0_{n,m}(\mathcal{V})$. Notice also that $\{0\}$ is always an orbit in $W^0_{n,m}(\mathcal{V})$, and every two words in the same orbit of the transversal $W_{n,m}(\mathcal{V})$ are similar.

Proposition 1.3 [13, Cor. 1]. The lattice of subvarieties of a hereditarily homogeneous semigroup variety $\mathcal{V}$ is anti-isomorphic to a subdirect product of lattices of the form $\mathrm{Con}(W^0_{n,m}(\mathcal{V}))$ taken over all n and all m such that $m \leqslant n$.

For referential convenience, we formulate, in an explicit form, a lemma that follows from [12, proof of Thm. 1.3].

LEMMA 1.7. Let m and n be natural numbers such that $m \leqslant n$ and $\mathcal{V}$ be an (n,m)-splittable nil-variety of semigroups. If α is a fully invariant congruence on a semigroup F corresponding to some subvariety of $\mathcal{V}$ then the restriction of α to a set $W^0_{n,m}(\mathcal{V})$ is a congruence of this $\mathbf{S}_m$-set. Conversely, every congruence of a $\mathbf{S}_m$-set $W^0_{n,m}(\mathcal{V})$ is a restriction to $W^0_{n,m}(\mathcal{V})$ of a fully invariant congruence on F corresponding to some subvariety in $\mathcal{V}$.

1.3. Multiplicative analogs of Proposition 1.3. Below is a (directly verifiable) lemma showing that in studying the multiplicative behavior of fully invariant congruences on relatively free semigroups, we can consider fully invariant congruences on a semigroup F. This will allow us to essentially simplify our further argument since manipulations with elements of F (i.e., ordinary semigroup words) are much easier than are with elements of relatively free semigroups.

LEMMA 1.8. Let $\mathcal{V}$ be a semigroup variety, ν a fully invariant congruence on a semigroup F corresponding to $\mathcal{V}$, and $r \in \overline{\mathbb{N}}$. The variety $\mathcal{V}$ is fi-r-permutable if and only if the fully invariant congruences on F containing ν are r-permutable.

Recall that a semigroup variety is said to be *locally nilpotent* if any one of its finitely generated semigroups is nilpotent. Repeated use will be made of the following three instrumental results on nil-semigroup identities (Lemma 1.9). The first of them is obvious, the second follows from [17, Lemma 1], and the third was proved in [9, Lemma 1.3(iii)].

LEMMA 1.9. Let $\mathcal{V}$ be a semigroup nil-variety. Then:

(a) If $\mathcal{V}$ satisfies an identity $u = v$ such that $c(u) \neq c(v)$ then it also satisfies $u = 0$.

(b) If $\mathcal{V}$ satisfies an identity of the form $x_1 x_2 \cdots x_n = v$, where $\ell(v) \neq n$, then it also satisfies $x_1 x_2 \cdots x_n = 0$.

(c) If $\mathcal{V}$ is locally nilpotent and satisfies an identity $u = v$ such that $\ell(u) < \ell(v)$ and $u \lhd v$ then it also satisfies $u = 0$.

A faithful multiplicative analog for Proposition 1.3 might be as follows: for $r \in \overline{\mathbb{N}}$, a hereditarily homogeneous variety $\mathcal{V}$ is fi-r-permutable iff all 0-transversals of the from $W^0_{n,m}(\mathcal{V})$ are congruence-r-permutable. This statement is invalid even for $r = 2$. (An example is easy to extract from [8, 9]; yet its somewhat weaker versions exist; cf. Prop. 1.4 and Lemma 1.11 below.)

If $\mathcal{V}$ is an (n, m)-splittable nil-variety, and α is a fully invariant congruence on a semigroup F corresponding to some subvariety of $\mathcal{V}$, then the restriction of α to $W^0_{n,m}(\mathcal{V})$ is denoted by $\alpha_{n,m}$. In view of Lemma 1.7, $\alpha_{n,m}$ is a congruence of the $\mathbf{S}_m$-set $W^0_{n,m}(\mathcal{V})$. In order to prove Proposition 1.4, we need the following:

LEMMA 1.10. Let m, n, and r be natural numbers such that $m \leqslant n$, $\mathcal{V}$ be a hereditarily (n, m)-splittable variety of nil-semigroups, and α and β be fully invariant congruences on F corresponding to some subvarieties of $\mathcal{V}$. Then $(u, v) \in \alpha_{n,m} \circ_r \beta_{n,m}$ if $u, v \in W^0_{n,m}(\mathcal{V})$ and $(u, v) \in \alpha \circ_r \beta$.

Proof. By the hypothesis, there exists a sequence $u_0, u_1, \ldots, u_r \in F$ of words such that $u_0 \equiv u$, $u_r \equiv v$, and for every $i = 0, 1, \ldots, r-1$, the pair (u_i, u_{i+1}) belongs to α, for even i, and to β for odd i. Since α and β correspond to subvarieties of $\mathcal{V}$, we may replace each one of $u_0, u_1, \ldots, u_r$ by its counterpart in $\mathcal{V}$. We therefore assume that $u_0, u_1, \ldots, u_r$ each belongs to some 0-transversal of the form $W^0_{k,\ell}(\mathcal{V})$. Specifically, for every $i = 0, 1, \ldots, r$, either $u_i \equiv 0$ or $u_i \neq 0$ in $\mathcal{V}$. If $u \equiv v$, the result is obvious. Assume that at least one of the words u and v is not equal to 0 in $\mathcal{V}$, say, $u \neq 0$ in $\mathcal{V}$, and hence $u \in W_{n,m}(\mathcal{V})$.

If $u_i \in W^0_{n,m}(\mathcal{V})$ for all $i = 0, 1, \ldots, r$ then $(u, v) \in \alpha_{n,m} \circ_r \beta_{n,m}$. Suppose now that there exists an i for which $u_i \notin W^0_{n,m}(\mathcal{V})$. Let i be least with this property. Clearly, $i > 0$. The pair (u_{i-1}, u_i) belongs to one of the congruences α and β, and $u_{i-1} \in W^0_{n,m}(\mathcal{V})$. This means that either $u_{i-1} \equiv 0$, or $c(u_{i-1}) \neq c(u_i)$, or $\ell(u_{i-1}) \neq \ell(u_i)$. In view of Lemma 1.9(a) and the fact that $\mathcal{V}$ is hereditarily (n, m)-splittable, a congruence, α or β, that contains a pair (u_{i-1}, u_i) also contains a pair $(u_{i-1}, 0)$.

A similar argument shows that if j is the greatest index such that $u_j \notin W^0_{n,m}(\mathcal{V})$, then $i \leqslant j < r$, and a congruence, α or β, that contains (u_j, u_{j+1}) also contains $(u_{j+1}, 0)$. In the sequence $u_0, u_1, \ldots, u_{i-1}, 0, u_{j+1}, \ldots, u_r$, each word belongs to $W^0_{n,m}(\mathcal{V})$, and every pair of its neighbors is contained either in α or in β. Consequently, every pair of neighboring words in this sequence belongs to $\alpha_{n,m}$, or to $\beta_{n,m}$. Keeping in mind that the length of the sequence in question does not exceed r, and $(u_0, u_1) \in \alpha$, we see that $(u,v)\in\alpha_{n,m}\circ_r\beta_{n,m}$.

Proposition 1.4. Let $\mathcal{V}$ be a semigroup nil-variety and m and n be natural numbers such that $m\leqslant n$ and $r\in\overline{\mathbb{N}}$. Then the $\mathbf{S}_m$-set $W^0_{n,m}(\mathcal{V})$ is congruence-r-permutable if $\mathcal{V}$ is hereditarily (n,m)-splittable and fi-r-permutable.

Proof. Denote by ν a fully invariant congruence on a semigroup F corresponding to $\mathcal{V}$. In view of Lemma 1.8, every two fully invariant congruences on F containing ν are r-permutable.

Let $\alpha,\beta\in\mathrm{Con}(W^0_{n,m}(\mathcal{V}))$. Our goal is to prove that α and β are r-permutable. From Lemma 1.7 and the fact that $\mu\in\mathrm{Con}(W^0_{n,m}(\mathcal{V}))$, it follows that $\mu=\overline{\mu}_{n,m}$ for some fully invariant congruence $\overline{\mu}$ on F containing ν.

First, assume that r is a natural number. By symmetry considerations, it suffices to prove that $\alpha\circ_r\beta\subseteq\beta\circ_r\alpha$. Let $u,v\in W^0_{n,m}(\mathcal{V})$ and $(u,v)\in\alpha\circ_r\beta$. Then $(u,v)\in\overline{\alpha}\circ_r\overline{\beta}$. Hence $(u,v)\in\overline{\beta}\circ_r\overline{\alpha}$. By Lemma 1.10, $(u,v)\in\beta\circ_r\alpha$.

Next, suppose that $r=s+0.5$ for some natural s. It suffices to verify that $\alpha\circ_{s+1}\beta\subseteq\alpha\circ_s\beta\cup\beta\circ_s\alpha$. Let $u,v\in W^0_{n,m}(\mathcal{V})$ and $(u,v)\in\alpha\circ_{s+1}\beta$. Clearly, $(u,v)\in\overline{\alpha}\circ_{s+1}\overline{\beta}$. Since congruences $\overline{\alpha}$ and $\overline{\beta}$ are $s.5$-permutable, we obtain $(u,v)\in\overline{\alpha}\circ_s\overline{\beta}\cup\overline{\beta}\circ_s\overline{\alpha}$, that is, either $(u,v)\in\overline{\alpha}\circ_s\overline{\beta}$ or $(u,v)\in\overline{\beta}\circ_s\overline{\alpha}$. Lemma 1.10 implies that $(u,v)\in\alpha\circ_s\beta$ in the former case and $(u,v)\in\beta\circ_s\alpha$ in the latter. Thus $(u,v)\in\alpha\circ_s\beta\cup\beta\circ_s\alpha$.

Recall that a semigroup variety is *permutable* if it satisfies a *permutation identity*, that is, an identity of the form

$$x_1x_2\cdots x_n=x_{1\pi}x_{2\pi}\cdots x_{n\pi}, \tag{1.1}$$

where $\pi\in\mathbf{S}_n$. The number n is called the *length* of identity (1.1). If $\mathcal{V}$ is a semigroup variety then $\mathrm{Perm}_n(\mathcal{V})$ denotes the set of all permutations $\pi\in\mathbf{S}_n$ for which identity (1.1) holds in $\mathcal{V}$. Clearly, $\mathrm{Perm}_n(\mathcal{V})$ is a subgroup in $\mathbf{S}_n$.

Proposition 1.4 entails

COROLLARY 1.1. Let $\mathcal{V}$ be a semigroup nil-variety, n be a natural number, and $r\in\overline{\mathbb{N}}$. If $W_{n,n}(\mathcal{V})\neq\varnothing$ then $W_{n,n}(\mathcal{V})$ is an $\mathbf{S}_n$-set which is congruence-r-permutable if and only if all groups in the interval $[\mathrm{Perm}_n(\mathcal{V}),\,\mathbf{S}_n]$ of a lattice $\mathrm{Sub}(\mathbf{S}_n)$ are r-permutable.

Proof. Put $W=W_{n,n}(\mathcal{V})$ and $W^0=W^0_{n,n}(\mathcal{V})$. From Lemma 1.9(b), it follows that every nil-semigroup variety is hereditarily (n,n)-splittable, and so W^0 and W are $\mathbf{S}_n$-sets. In view of Proposition 1.4, the $\mathbf{S}_n$-set W^0 is congruence-r-permutable. Consequently, its $\mathbf{S}_n$-subset W too is congruence-r-permutable. Clearly, W is transitive. It is easy to see that $\mathrm{Stab}_W(x_1x_2\cdots x_n)=\mathrm{Perm}_n(\mathcal{V})$ (cf. [9, proof of Cor. 1.7]). It remains to appeal to Lemma 1.4.

From Proposition 1.4 it follows, in particular, that if $\mathcal{V}$ is a hereditarily homogeneous variety, and $r\in\overline{\mathbb{N}}$, then the fact that $\mathcal{V}$ is fi-r-permutable implies that all 0-transversals of the form $W^0_{n,m}(\mathcal{V})$ are congruence-r-permutable. The converse of this statement is invalid in general, but is valid if $r\geqslant 2.5$ and all non-empty transversals of the form $W_{n,m}(\mathcal{V})$ are transitive (neither condition can be discarded). For $r\in\{2.5,3\}$, the following lemma holds.

LEMMA 1.11. Assume that $r\in\{2.5,3\}$, $\mathcal{V}$ is a hereditarily homogeneous semigroup variety, and the conditions below are satisfied:

(a) all 0-transversals like $W^0_{n,m}(\mathcal{V})$ are congruence-r-permutable;

(b) at most one non-empty transversal like $W_{n,m}(\mathcal{V})$ is not transitive;

(c) if $W_{n,m}(\mathcal{V})\neq\varnothing$, $W_{n,m}(\mathcal{V})$ is not transitive, $w\neq 0$ in $\mathcal{V}$, and $w\notin F_{n,m}(\mathcal{V})$, then either w divides in $\mathcal{V}$ some word in $W_{n,m}(\mathcal{V})$, or any word in $W_{n,m}(\mathcal{V})$ divides in $\mathcal{V}$ a word w.

Then $\mathcal{V}$ is fi-r-permutable.

Proof. Let ν be a fully invariant congruence on F corresponding to $\mathcal{V}$ and α and β be fully invariant congruences on F containing ν. In view of Lemma 1.8, it suffices to verify that α and β are r-permutable.

Let $u, v \in F$ and $(u, v) \in \alpha\beta\alpha$, that is, $u\,\alpha\,w_1\,\beta\,w_2\,\alpha\,v$ for some words w_1 and w_2. We have to prove that $(u, v) \in \alpha\beta \cup \beta\alpha$, if $r = 2.5$, and $(u, v) \in \beta\alpha\beta$ if $r = 3$. Since $\alpha, \beta \supseteq \nu$, all words u, w_1, w_2, and v can be replaced by their counterparts in $\mathcal{V}$. We may therefore assume that u, w_1, w_2, and v each lies in some 0-transversal $W^0_{n,m}(\mathcal{V})$, and either coincides with 0 or is not equal to 0 in $\mathcal{V}$. Again u, w_1, w_2, and v may be thought of as being pairwise distinct, since otherwise it is obvious that $(u, v) \in \alpha\beta \cup \beta\alpha \subseteq \beta\alpha\beta$. In particular, one of the words u and v is not equal to 0 in $\mathcal{V}$. There is no loss of generality in assuming that $u \neq 0$ in $\mathcal{V}$, and hence $u \in W_{n,m}(\mathcal{V})$ for some n and m.

In view of (a), the 0-transversal $W^0_{n,m}(\mathcal{V})$ is congruence-r-permutable. Proposition 1.2 implies that it contains at most three orbits, and hence $W_{n,m}(\mathcal{V})$ contains at most two. Specifically, if such is not transitive then it contains exactly two orbits. There are six cases to consider.

Case 1. Let $w_1, w_2, v \in W^0_{n,m}(\mathcal{V})$. Then $(u, v) \in \alpha_{n,m}\beta_{n,m}\alpha_{n,m}$. Condition (a) implies that $(u, v) \in \alpha_{n,m}\beta_{n,m} \cup \beta_{n,m}\alpha_{n,m} \subseteq \alpha\beta \cup \beta\alpha$, if $r = 2.5$, and $(u, v) \in \beta_{n,m}\alpha_{n,m}\beta_{n,m} \subseteq \beta\alpha\beta$ if $r = 3$.

Case 2. Let $w_1, w_2 \in W_{n,m}(\mathcal{V})$ and let $v \in W_{k,\ell}(\mathcal{V})$ for some transversal $W_{k,\ell}(\mathcal{V})$ distinct from $W_{n,m}(\mathcal{V})$. This, by virtue of the fact that $\mathcal{V}$ is hereditarily homogeneous, implies that the congruence α contains pairs $(w_2, 0)$ and $(v, 0)$. If $W_{n,m}(\mathcal{V})$ is transitive then $u \approx w_2$. Therefore, along with $(w_2, 0)$, α contains $(u, 0)$, whence $u\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$.

Now, suppose that $W_{n,m}(\mathcal{V})$ is not transitive. The above argument implies that it contains exactly two orbits. On the Dirichlet principle, at least two of the words u, w_1, and w_2 lie in the same orbit and are similar. If $w_1 \approx w_2$ then, along with $(w_2, 0)$, α contains $(w_1, 0)$. Therefore $u\,\alpha\,w_1\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$. Similarly, if $u \approx w_2$ then α contains $(u, 0)$ along with $(w_2, 0)$, and so $u\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$. Now, let $u \approx w_1$. Recall that $w_1\,\beta\,w_2$. Consequently, $u\,\beta\,w_2'$ for some word w_2' such that $w_2' \approx w_2$. Then α contains $(w_2', 0)$ along with $(w_2, 0)$, and so $u\,\beta\,w_2'\,\alpha\,0\,\alpha\,v$, that is, $(u, v) \in \beta\alpha$.

Case 3. Let $w_1 \in W_{n,m}(\mathcal{V})$ and $w_2 \equiv 0$. If $v \in W^0_{n,m}(\mathcal{V})$ then we arrive at Case 1. Therefore we may assume that $v \in W_{k,\ell}(\mathcal{V})$ for some transversal $W_{k,\ell}(\mathcal{V})$ distinct from $W_{n,m}(\mathcal{V})$. If $W_{n,m}(\mathcal{V})$ is transitive then $u \approx w_1$. Consequently, together with $(w_1, 0)$, β contains $(u, 0)$. Hence $u\,\beta\,0\,\alpha\,v$, that is, $(u, v) \in \beta\alpha$. Similarly we treat the situation where $W_{n,m}(\mathcal{V})$ is not transitive and words u and w_1 lie in the same orbit.

Now, suppose that $W_{n,m}(\mathcal{V})$ is not transitive, while u and w_1 lie in its distinct orbits. The above argument implies that $W_{n,m}(\mathcal{V})$ contains exactly two orbits. By (b), either v divides in $\mathcal{V}$ some word in $W_{n,m}(\mathcal{V})$, or any word in $W_{n,m}(\mathcal{V})$ divides in $\mathcal{V}$ a word v. First, assume that v divides in $\mathcal{V}$ some word w in $W_{n,m}(\mathcal{V})$. Then v divides in $\mathcal{V}$ any word in that orbit of $W_{n,m}(\mathcal{V})$ which contains w. Since this transversal contains just two orbits, and u and w_1 belong to distinct orbits, we conclude that v divides in $\mathcal{V}$ one of the words u or w_1. If $v \overset{\mathcal{V}}{\lhd} u$, then α contains $(u, 0)$ along with $(v, 0)$, whence $u\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$. If $v \overset{\mathcal{V}}{\lhd} w_1$, then α contains $(w_1, 0)$ together with $(v, 0)$. Consequently, $u\,\alpha\,w_1\,\alpha\,0\,\alpha\,v$, that is, again $u\,\alpha\,v$. It remains to handle the situation where any word in $W_{n,m}(\mathcal{V})$ divides in $\mathcal{V}$ a word v. In particular, $w_1 \overset{\mathcal{V}}{\lhd} v$. Consequently, β contains $(v, 0)$ along with $(w_1, 0)$. It follows that $u\,\alpha\,w_1\,\beta\,0\,\beta\,v$, that is, $(u, v) \in \alpha\beta$.

Case 4. Let $w_1 \in W_{n,m}(\mathcal{V})$ and let $w_2 \in W_{k,\ell}(\mathcal{V})$ for some transversal $W_{k,\ell}(\mathcal{V})$ distinct from $W_{n,m}(\mathcal{V})$. This, in view of the fact that $\mathcal{V}$ is hereditarily homogeneous, implies that β contains pairs $(w_1, 0)$ and $(w_2, 0)$. If $v \equiv 0$ then $u\,\alpha\,w_1\,\beta\,v$, that is, $(u, v) \in \alpha\beta$. If v belongs to some transversal other than $W_{k,\ell}(\mathcal{V})$, then α contains a pair $(v, 0)$, since $\mathcal{V}$ is hereditarily homogeneous. Thus $u\,\alpha\,w_1\,\beta\,0\,\alpha\,v$, and we are in the situation considered in Case 3. Lastly, let $v \in W_{k,\ell}(\mathcal{V})$. By (b), at least one of the transversals $W_{n,m}(\mathcal{V})$

and $W_{k,\ell}(\mathcal{V})$ is transitive. If $W_{n,m}(\mathcal{V})$ is transitive then $u \approx w_1$. It follows that β contains $(u, 0)$ along with $(w_1, 0)$, whence $u\,\beta\,0\,\beta\,w_2\,\alpha\,v$, that is, $(u, v) \in \beta\alpha$. If $W_{k,\ell}(\mathcal{V})$ is transitive then $v \approx w_2$. This implies that β contains $(v, 0)$ together with $(w_2, 0)$, whence $u\,\alpha\,w_1\,\beta\,0\,\beta\,v$, that is, $(u, v) \in \alpha\beta$.

Case 5. Let $w_1 \equiv 0$. We have $w_2, v \not\equiv 0$. If w_2 and v lie in the same transversal then we arrive at the situation dual to Case 3. Otherwise, α contains $(v, 0)$ since $\mathcal{V}$ is hereditarily homogeneous. It follows that $u\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$.

Case 6. Let $w_1 \in W_{k,\ell}(\mathcal{V})$ for some transversal $W_{k,\ell}(\mathcal{V})$ distinct from $W_{n,m}(\mathcal{V})$. This, by virtue ot the fact that $\mathcal{V}$ is hereditarily homogeneous, implies that α contains $(u, 0)$. If $v \equiv 0$ then $u\,\alpha\,v$. If $w_2 \equiv 0$ then $u\,\alpha\,w_2\,\alpha\,v$, that is, again $u\,\alpha\,v$. If w_2 and v belong to the same transversal then we arrive at the situation dual to Case 2 (if $w_2, v \in W_{k,\ell}(\mathcal{V})$), or to Case 4 (if $w_2, v \notin W_{k,\ell}(\mathcal{V})$). We may therefore assume that w_2 and v belong to distinct transversals. Then the property of being hereditarily homogeneous for $\mathcal{V}$ insists on α containing $(v, 0)$. Consequently, $u\,\alpha\,0\,\alpha\,v$, that is, $u\,\alpha\,v$.

2. PROOF OF THE THEOREM: NECESSITY

Denote by $\mathcal{ZM}$ the variety of all semigroups with zero multiplication. Recall that a semigroup variety is said to be *completely regular* if its semigroups each is a union of groups. The next two lemmas are generalizations of Lemmas 1.6 and 1.5 in [8], respectively.

LEMMA 2.1. If a semigroup variety is weakly fi-permutable then either it is completely regular or is a nil-variety.

Proof. It is known that $\mathcal{ZM}$ is an atom in the lattice of all semigroup varieties, and that a semigroup variety is completely regular (is a nil-variety) iff it does not contain $\mathcal{ZM}$ (whenever the lattice of its subvarieties does not contain atoms other than $\mathcal{ZM}$); see, e.g., [18].

Let $\mathcal{V}$ be a weakly fi-permutable variety of semigroups. Suppose $\mathcal{V} \supseteq \mathcal{ZM}$. It suffices to verify that $L(\mathcal{V})$ lacks atoms other than $\mathcal{ZM}$. Assume, to the contrary, that $L(\mathcal{V})$ contains an atom $\mathcal{A}$ distinct from $\mathcal{ZM}$ and that α and ζ are fully invariant congruences on a semigroup F corresponding to the varieties $\mathcal{A}$ and $\mathcal{ZM}$, respectively. Since $\mathcal{ZM} \wedge \mathcal{A}$ is a trivial variety, the fully invariant congruence $\zeta \vee \alpha$ coincides with the universal relation ∇ on F. By Lemma 1.8, ζ and α are weakly permutable, and so $\zeta\alpha\zeta = \nabla$. In particular, $(x, y) \in \zeta\alpha\zeta$ for every two distinct letters x and y, that is, $x\,\zeta\,u\,\alpha\,v\,\zeta\,y$ for some words u and v. Identities $x = u$ and $v = y$ hold in $\mathcal{ZM}$, whence $u \equiv x$ and $v \equiv y$. Then $u\,\alpha\,v$ implies that $\mathcal{A}$ satisfies $x = y$, contrary to the choice of $\mathcal{A}$.

LEMMA 2.2. If a completely regular semigroup variety is fi-2.5-permutable then either it is completely simple or coincides with $\mathcal{SL}$.

Proof. It is known that a completely regular variety of semigroups is completely simple iff it does not contain the variety $\mathcal{SL}$ (which is an atom in the lattice of all semigroup varieties), and that the lattice of subvarieties of every variety strictly containing $\mathcal{SL}$ contains an atom distinct from $\mathcal{SL}$; see, e.g., [18].

Let $\mathcal{V}$ be an fi-2.5-permutable completely regular semigroup variety and $\mathcal{V} \supseteq \mathcal{SL}$. It suffices to verify that $L(\mathcal{V})$ lacks atoms other than $\mathcal{SL}$. Assume, to the contrary, that $L(\mathcal{V})$ contains, along with $\mathcal{SL}$, yet another atom $\mathcal{A}$, and that α and σ are fully invariant congruences on a semigroup F corresponding to the varieties $\mathcal{A}$ and $\mathcal{SL}$, respectively. Since the variety $\mathcal{SL} \wedge \mathcal{A}$ is trivial, its corresponding fully invariant congruence $\sigma \vee \alpha$ coincides with ∇ on F.

By Lemma 1.8, σ and α are 2.5-permutable, and so $\sigma\alpha \cup \alpha\sigma = \nabla$. In particular, $(x, y) \in \sigma\alpha \cup \alpha\sigma$ for every two distinct letters x and y. First, suppose that $(x, y) \in \sigma\alpha$, that is, $x\,\sigma\,u\,\alpha\,y$ for some word u. It is easy to verify that $u = v$ holds in $\mathcal{SL}$ iff $c(u) = c(v)$. Since $x \, \sigma \, u$, we have $c(u) = \{x\}$, that is, $u \equiv x^n$ for some n. It follows from $u \, \alpha \, y$ that $\mathcal{A}$ satisfies an identity $x^n = y$. If we substitute x for y in this identity we obtain $x^n = x$, and so $x = y$, which contradicts the choice of $\mathcal{A}$. The case $(x, y) \in \alpha\sigma$ can be verified similarly.

Now, let $\mathcal{V}$ be an fi-2.5-permutable but not completely simple semigroup variety and $\mathcal{V} \neq \mathcal{SL}$. In view of Lemmas 2.1 and 2.2, $\mathcal{V}$ is a nil-variety. Our goal is to prove that $\mathcal{V}$ satisfies one of the systems (0.3)-(0.22), where in (0.3)-(0.8) and (0.17)-(0.19), π has the meaning specified in the formulation of the theorem. To do this, we need a number of lemmas.

LEMMA 2.3. If a semigroup nil-variety is fi-2.5-permutable then it satisfies a non-trivial permutation identity of length 3.

Proof. Let $\mathcal{V}$ be an fi-2.5-permutable nil-variety of semigroups. We may assume that $W_{3,3}(\mathcal{V}) \neq \varnothing$; otherwise, $\mathcal{V}$ satisfies any permutation identity of length 3. Suppose that the conclusion of the lemma is invalid. Then $\mathrm{Perm}_3(\mathcal{V})$ is a trivial group, and the interval $[\mathrm{Perm}_3(\mathcal{V}) \, \mathbf{S}_3]$ of a lattice $\mathrm{Sub}(\mathbf{S}_3)$ coincides with the entire lattice. In view of Lemma 1.9(b), $\mathcal{V}$ is hereditarily (3.3)-splittable. Proposition 1.4 implies that the $\mathbf{S}_3$-set $W^0_{3,3}(\mathcal{V})$ is congruence-2.5-permutable, and so its $\mathbf{S}_3$-subset $W_{3,3}(\mathcal{V})$ is likewise. By Corollary 1.1, all subgroups of $\mathbf{S}_3$ are 2.5-permutable, a contradiction with Lemma 1.5.

It is well known that every permutation semigroup nil-variety is locally nilpotent. Therefore Lemma 2.3 allows item (c) of Lemma 1.9 to be applied throughout the remaining part of this section, which we will do without further comment.

In [9, proof of Lemma 2.8], the following was proved:

LEMMA 2.4. If a semigroup nil-variety satisfies a non-trivial permutation identity of length 3 then it is (3,2)-splittable.

LEMMA 2.5. Let $\mathcal{V}$ be an fi-2.5-permutable semigroup nil-variety. Then:

(a) $\mathcal{V}$ satisfies one of the identities

$$x^2y = y^2x, \tag{2.1}$$

$$xy^2 = yx^2, \tag{2.2}$$

$$x^2y = xy^2, \tag{2.3}$$

$$x^2y = yx^2. \tag{2.4}$$

(b) $\mathcal{V}$ satisfies either identity (2.1) or one of the following:

$$xyx = yxy, \tag{2.5}$$

$$x^2y = xyx, \tag{2.6}$$

$$x^2y = yxy. \tag{2.7}$$

Proof. A proof for (a) and (b) follows the same line of argument, and we so limit ourselves to verifying the former. In view of Lemma 2.3, $\mathcal{V}$ satisfies a non-trivial permutation identity of length 3. By Lemma 2.4, $\mathcal{V}$ is hereditarily (3,2)-permutable. In view of Propositions 1.4 and 1.2, the $\mathbf{S}_2$-set $W^0_{3,2}(\mathcal{V})$ is congruence-2.5-permutable and segregated. If $\mathcal{V}$ does not satisfy any one of (2.1)-(2.4) then $W^0_{3,2}(\mathcal{V})$ contains two isomorphic non one-element orbits — $\{x^2y, y^2x\}$ and $\{xy^2, yx^2\}$, which is impossible by Lemma 1.1.

LEMMA 2.6. Let $\mathcal{V}$ be a hereditarily (n, m)-splittable weakly fi-permutable semigroup nil-variety. If the set $W_{n,m}(\mathcal{V})$ is non-empty then either the transversal $W_{n,m}(\mathcal{V})$ is transitive, or $\mathcal{V}$ satisfies an identity $x_1x_2\cdots x_{n+1} = 0$.

Proof. Since all non-empty transversals of the form $W_{n,n}(\mathcal{V})$ are transitive, we may assume that $m < n$. For brevity, put $W = W_{n,m}(\mathcal{V})$ and $W^0 = W^0_{n,m}(\mathcal{V})$. Suppose that $\mathcal{V}$ fails to satisfy $x_1x_2\cdots x_{n+1} = 0$, $W \neq \varnothing$, and W is not a transitive transversal. In virtue of Propositions 1.4 and 1.2, the 0-transversal W^0 contains at most three orbits. Consequently, W contains at most two, and being non-transitive, exactly two orbits. Let u and v be words in distinct orbits of this transversal. Clearly, $\mathcal{V}$ does not satisfy any one of $u = v$, $u = 0$, and $v = 0$.

Consider subvarieties $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{V}$, the first of which is defined in $\mathcal{V}$ by identities $u = v$ and $x_1x_2\cdots x_{n+1} = 0$, and the second — by $v = 0$. By the choice of u and v, then, identities $u = 0$ and $v = 0$ do not hold in $\mathcal{A}$, and nor do $u = 0$ and $x_1x_2\cdots x_{n+1} = 0$ in $\mathcal{B}$. Denote by α and β fully invariant congruences on a semigroup F corresponding to the varieties $\mathcal{A}$ and $\mathcal{B}$, respectively. In view of Lemma 1.8, α and β are weakly permutable. Since $u\,\alpha\,v\,\beta\,0\,\alpha\,x_1x_2\cdots x_{n+1}$, we obtain $(u, x_1x_2\cdots x_{n+1}) \in \alpha\beta\alpha$. Consequently, $(u, x_1x_2\cdots x_{n+1}) \in \beta\alpha\beta$. In other words, there exist words w_1 and w_2 such that $u\,\beta\,w_1\,\alpha\,w_2\,\beta\,x_1x_2\cdots x_{n+1}$.

If $w_2 \not\approx x_1x_2\cdots x_{n+1}$ then the fact that $\mathcal{B}$ satisfies $w_2 = x_1x_2\cdots x_{n+1}$ and Lemma 1(b) imply that $x_1x_2\cdots x_{n+1} = 0$ in $\mathcal{B}$. Consequently, $w_2 \approx x_1x_2\cdots x_{n+1}$. If either $\ell(w_1) \neq n$ or $c(w_1) \neq \{x_1, x_2, \ldots, x_m\}$, then the property of $\mathcal{V}$ being hereditarily (n, m)-splittable, Lemma 1.9(a), and the fact that $u = w_1$ in $\mathcal{B}$ entail that $\mathcal{B}$ satisfies $u = 0$, which is impossible. Finally, let $\ell(w_1) = n$ and $c(w_1) = \{x_1, x_2, \ldots x_m\}$. Recall that $w_2 \approx x_1x_2\cdots x_{n+1}$. From Lemma 1.9(b) and the fact that $\mathcal{A}$ satisfies $w_1 = w_2$, it follows that $w_1 = 0$ in $\mathcal{A}$. First, let $w_1 \neq 0$ in $\mathcal{V}$. Then $w_1 \in F_{n,m}(\mathcal{V})$, and so w_1 is equal in $\mathcal{V}$ to some word in W. Since W contains exactly two orbits, and u and v lie in its distinct orbits, we see that each word in W is similar to one of the words u and v. That is, either $w_1 \overset{\mathcal{V}}{\approx} u$ or $w_1 \overset{\mathcal{V}}{\approx} v$. Consequently, $\mathcal{A}$ satisfies one of the identities $u = 0$ and $v = 0$, a contradiction. Lastly, let $w_1 = 0$ in $\mathcal{V}$. Then $w_1 = 0$ in $\mathcal{B}$, and hence $u = 0$ in $\mathcal{B}$, which is impossible.

LEMMA 2.7. Let $\mathcal{V}$ be a weakly fi-permutable semigroup nil-variety. Then:

(a) If $\mathcal{V}$ satisfies one of the identities $xyz = yxz$ and $xyz = xzy$ then $\mathcal{V}$ also satisfies either (2.3) or one of the following:

$$x^2y = 0, \tag{2.8}$$

$$xy^2 = 0, \tag{2.9}$$

$$xyzt = 0. \tag{2.10}$$

(b) If $\mathcal{V}$ satisfies $xyz = zyx$ then $\mathcal{V}$ also satisfies one of (2.6), (2.8), (2.10) or the following:

$$xyx = 0. \tag{2.11}$$

Proof. (a) We may assume that $W_{3,2}(\mathcal{V}) \neq \varnothing$, for otherwise $\mathcal{V}$ satisfies any one of (2.3), (2.8), and (2.9). By Lemmas 2.4 and 2.6, either $\mathcal{V}$ satisfies (2.10), or the transversal $W_{3,2}(\mathcal{V})$ is transitive. If $\mathcal{V}$ does not satisfy any one of (2.3), (2.8), and (2.9) then the words x^2y and xy^2 belong to different orbits of $W_{3,2}(\mathcal{V})$.

(b) Is verified similarly.

For every $i = 1, 2, \ldots, n$, put $\mathrm{Stab}_n(i) = \{\pi \in \mathbf{S}_n \mid i\pi = i\}$. Clearly, $\mathrm{Stab}_n(i)$ is a subgroup of $\mathbf{S}_n$. From [19] we infer the following:

LEMMA 2.8. Let $\mathcal{V}$ be a semigroup variety satisfying a non-trivial permutation identity of length 3. If $n \geqslant 4$ then the group $\mathrm{Perm}_n(\mathcal{V})$ contains one of the groups $\mathrm{Stab}_n(1)$ and $\mathrm{Stab}_n(n)$.

LEMMA 2.9. If a semigroup nil-variety is fi-2.5-permutable then it is $(4,2)$-splittable.

Proof. Let $\mathcal{V}$ be an fi-2.5-permutable nil-variety of semigroups. By Lemma 2.3, $\mathcal{V}$ satisfies a non-trivial permutation identity of length 3, and so we may apply Lemma 1.9(c). Suppose that $\mathcal{V}$ satisfies an identity $u = v$, where $\ell(u) = 4$, $n(u) = 2$, and $\ell(v) > 4$. In view of Lemma 1.9(a), we can put $c(u) = c(v)$. By Lemma 2.8, u is similar in $\mathcal{V}$ to one of the words x^3y, xy^3, and x^2y^2, and we may assume that u coincides with one of these. In particular, $c(u) = c(v) = \{x, y\}$.

Put $k = \ell_x(v)$ and $\ell = \ell_y(v)$. If $k \geqslant 4$ or $\ell \geqslant 4$ then $u \lhd v$ and $u = 0$ in $\mathcal{V}$ by Lemma 1.9(c). Since $\ell(v) \geqslant 5$, either $k = \ell = 3$, or $k = 3$ and $\ell = 2$, or $k = 2$ and $\ell = 3$. By Lemma 2.8, v is equal in $\mathcal{V}$ to one of the words x^3y^3, y^3x^3, x^3y^2, y^3x^2, x^2y^3, and y^2x^3, and we may assume that v coincides with one of these. From Lemma 1.9(c), the result follows immediately provided that either $u \equiv x^2y^2$, or $v \in \{x^3y^3, y^3x^3\}$, or $u \equiv x^3y$ and $v \in \{x^3y^2, y^3x^2\}$, or $u \equiv xy^3$ and $v \in \{x^2y^3, y^2x^3\}$.

We are left with the following two possibilities: $u \equiv x^3y$ and $v \in \{x^2y^3, y^2x^3\}$ or $u \equiv xy^3$ and $v \in \{x^3y^2, y^3x^2\}$. By Lemma 2.5(a), $\mathcal{V}$ satisfies one of (2.1)-(2.4). First, assume that one of (2.1)-(2.3) holds in $\mathcal{V}$. Substituting x^2 for x in each of these identities, we see that $\mathcal{V}$ satisfies one of $x^4y = y^2x^2$, $x^2y^2 = yx^4$, or $x^4y = x^2y^2$. In any case Lemma 1.9(c) implies that $x^2y^2 = 0$ in $\mathcal{V}$, and so $v = 0$ holds in $\mathcal{V}$.

Now, let $\mathcal{V}$ satisfy (2.4). If $u \equiv x^3y$ and $v \in \{x^2y^3, y^2x^3\}$ then v is equal in $\mathcal{V}$ to one of the words y^3x^2 and x^3y^2, and so $u \overset{\mathcal{V}}{\lhd} v$. By Lemma 1.9(c), we have $u = 0$ in $\mathcal{V}$. The case where $u \equiv xy^3$ and $v \in \{x^3y^2, y^3x^2\}$ can be treated similarly.

LEMMA 2.10. If a semigroup nil-variety is fi-2.5-permutable then it satisfies one of the identities

$$x^3y = x^2y^2, \tag{2.12}$$

$$x^3y = 0, \tag{2.13}$$

$$x^2y^2 = 0, \tag{2.14}$$

$$x_1x_2x_3x_4x_5 = 0. \tag{2.15}$$

Proof. Let $\mathcal{V}$ be an fi-2.5-permutable nil-variety of semigroups. We may assume that $W_{4,2}(\mathcal{V}) \neq \varnothing$, for otherwise $\mathcal{V}$ satisfies one of (2.12)-(2.14). In view of Lemmas 2.6 and 2.9, either $\mathcal{V}$ satisfies (2.15), or the transversal $W_{4,2}(\mathcal{V})$ is transitive. If $\mathcal{V}$ does not satisfy any one of (2.12)-(2.14) then the words x^3y and x^2y^2 belong to different orbits of $W_{4,2}(\mathcal{V})$.

LEMMA 2.11 [9, Lemma 2.12]. Let $\mathcal{V}$ be a semigroup nil-variety satisfying a non-trivial permutation identity of length 3. Then:

(a) If $\mathcal{V}$ satisfies one of (2.3), (2.7) then it also satisfies $x^2yz = 0$.

(b) If $\mathcal{V}$ satisfies (2.12) then it also satisfies $x^2y^2z = 0$.

We finish to prove that the hypothesis of the theorem is necessary. Let $\mathcal{V}$ be an fi-2.5-permutable nil-variety of semigroups. In view of Lemma 2.3, $\mathcal{V}$ satisfies an identity of the form $x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi}$, where π is one of the permutations (12), (13), (23), or (123).

First, suppose that π is one of (12), (23). In view of Lemma 2.5(a) [Lemma 2.7(a)], $\mathcal{V}$ satisfies one of (2.1)-(2.4) [one of (2.3), (2.8)-(2.10)]. If one of (2.8), (2.9) holds in $\mathcal{V}$ then we are faced up to one of the systems (0.3) and (0.4), respectively. If $\mathcal{V}$ satisfies (2.3) then it satisfies system (0.5) by Lemma 2.11(a). If $\mathcal{V}$ satisfies (2.10) and one of (2.1), (2.2) then we arrive at one of the systems (0.17) and (0.18), respectively. Let $\mathcal{V}$ satisfy (2.4). In virtue of Lemma 2.10, $\mathcal{V}$ satisfies one of (2.12)-(2.15). If it satisfies one of (2.13)-(2.15) then we obtain one of the systems (0.6), (0.7), and (0.19), respectively. If (2.12) holds in $\mathcal{V}$ then $\mathcal{V}$ satisfies system (0.8) by Lemma 2.11(b).

Next, assume that $\pi = (13)$. By Lemma 2.5(b) [Lemma 2.7(a)], $\mathcal{V}$ satisfies one of (2.1), (2.5)-(2.7) [one of (2.6), (2.8), (2.10), and (2.11)]. If one of (2.8) and (2.11) holds in $\mathcal{V}$ then we obtain one of the systems (0.3) and (0.9), respectively. If (2.7) holds in $\mathcal{V}$ then $\mathcal{V}$ satisfies system (0.10) by Lemma 2.11(a). If, however, (2.10) and one of (2.1), (2.5) hold in $\mathcal{V}$ then we arrive at one of the systems (0.17) and (0.20), respectively. Suppose that $\mathcal{V}$ satisfies (2.6). In view of Lemma 2.10, $\mathcal{V}$ satisfies one of (2.12)-(2.15). If $\mathcal{V}$ satisfies one of (2.13)-(2.15) then we arrive at one of the systems (0.11), (0.12), and (0.21), respectively. If (2.12) holds in $\mathcal{V}$ then $\mathcal{V}$ satisfies system (0.13) by Lemma 2.11(b).

Finally, let $\pi = (123)$. In view of Lemma 2.10, $\mathcal{V}$ satisfies one of (2.12)-(2.15). If $\mathcal{V}$ satisfies one of (2.13)-(2.15) then we are faced up to one of the systems (0.14), (0.15), and (0.22), respectively. And if $\mathcal{V}$ satisfies (2.12) then $\mathcal{V}$ satisfies system (0.16) by Lemma 2.11(b). The necessity is proved.

3. PROOF OF THE THEOREM: SUFFICIENCY

In [4, 5] it was proved that every completely simple variety is fi-permutable. Since $\mathcal{SL}$ is an atom in the lattice of all semigroup varieties, every $\mathcal{SL}$-free semigroup has at most two fully invariant congruences. Clearly, these are permutable. Therefore it remains to consider the case where $\mathcal{V}$ satisfies one of the systems (0.3)-(2.22), where in (0.3)-(0.8) and (0.17)-(0.19), π has the meaning specified in the formulation of the theorem. Our goal is to prove that $\mathcal{V}$ is fi-2.5-permutable.

First, assume that $\mathcal{V}$ satisfies one of (0.3)-(0.16). From [9, Lemma 3.4], it follows that in this case $\mathcal{V}$ is hereditarily homogeneous, all 0-transversals of the form $W^0_{n,m}(\mathcal{V})$ are congruence-permutable, and all non-empty transversals of the form $W_{n,m}(\mathcal{V})$ are transitive. By Lemma 1.11, $\mathcal{V}$ is fi-2.5-permutable. It remains to handle the varieties specified by systems (0.17)-(0-22).

LEMMA 3.1. Let $\mathcal{V}$ be a semigroup variety specified by one of the systems (0.17)-(0.22), where in (0.17)-(0.19), π is as in the formulation of the theorem. Then $\mathcal{V}$ is hereditarily homogeneous, all 0-transversals like $W^0_{n,m}(\mathcal{V})$ are segregated and contain at most three orbits, and congruence lattices of all orbits of these 0-transversals contain at most two elements.

Proof. All non-empty transversals of the form $W_{n,m}(\mathcal{V})$, for $1 < m < n$, are given in the second column of Table 1; semicolon separates orbits of the transversals. Using Table 1 and Lemma 1.9, it is easy to state that $\mathcal{V}$ is hereditarily homogeneous.

Now, let $m, n \in \mathbb{N}$ and $m \leqslant n$. If $W_{n,m}(\mathcal{V}) = \varnothing$, then $W^0_{n,m}(\mathcal{V}) = \{0\}$, and so all the required statements are obvious. We may therefore assume that $W_{n,m}(\mathcal{V}) \neq \varnothing$. If $m = 1$ then the result is again apparent since the 0-transversal $W^0_{n,1}(\mathcal{V})$ consists of two one-element orbits — $\{x_1^n\}$ and $\{0\}$.

Let $1 < m < n$. Looking at Table 1 we see that in this event all non-empty transversals of the form $W_{n,m}(\mathcal{V})$ contain at most two orbits, and so all 0-transversals of the form $W^0_{n,m}(\mathcal{V})$ contain at most three. Table 1 and Lemma 1.2 entail that all 0-transversals $W^0_{n,m}(\mathcal{V})$ are segregated. Lastly, Table 1 and Lemma 1.6 imply that congruence lattices of all orbits of these 0-transversals contain at most two elements.

Let $m = n$. It is clear that the $\mathbf{S}_n$-set $W^0_{n,n}(\mathcal{V})$ contain two orbits — $W_{n,n}(\mathcal{V})$ and $\{0\}$. That the 0-transversal $W^0_{n,n}(\mathcal{V})$ is segregated now follows from Lemma 1.2. We claim that the transversal $W_{n,n}(\mathcal{V})$ contains at most two congruences. In view of Corollary 1.1, it suffices to verify that the interval $[\mathrm{Perm}_n(\mathcal{V}),\ \mathbf{S}_n]$ of $\mathrm{Sub}(\mathbf{S}_n)$ hosts at most two elements. For $n \leqslant 2$, this is obvious since the entire lattice $\mathrm{Sub}(\mathbf{S}_n)$ contains at most two elements. Now, let $n = 3$. Systems (0.17)-(0.22) each have a non-trivial permutation identity

TABLE 1. Non-Empty Transversals

<table>
<tr><th>$\mathcal{V}$ is specified by one of the systems</th><th>Non-empty transversals of the form $W_{n,m}(\mathcal{V})$, where $1 < m < n$</th><th>Words in transitive transversals (up to similarity)</th></tr>
<tr><td>(0.17) for $\pi \in \{(12), (23)\}$</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y; xy^2, yx^2\}$</td><td>$x$, x^2, x^3, xy, xyz</td></tr>
<tr><td>(0.17) for $\pi = (13)$</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y; xyx, yxy\}$</td><td>$x$, x^2, x^3, xy, xyz</td></tr>
<tr><td>(0.18) for $\pi \in \{(12), (23)\}$</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y, y^2x; xy^2\}$</td><td>$x$, x^2, x^3, xy, xyz</td></tr>
<tr><td rowspan="3">(0.19) for $\pi = (12)$,
(0.21),
(0.22)</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y, y^2x\}$</td><td rowspan="3">$x$, x^2, x^3, x^4,
xy, xyz, $xyzt$,
x^2y, x^2yz</td></tr>
<tr><td>$W_{4,2}(\mathcal{V}) = \{x^3y, y^3x; x^2y^2\}$</td></tr>
<tr><td>$W_{4,3}(\mathcal{V}) = \{x^2yz, y^2xz, z^2xy\}$</td></tr>
<tr><td rowspan="3">(0.19) for $\pi = (23)$</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y, y^2x\}$</td><td rowspan="3">$x$, x^2, x^3, x^4,
xy, xyz, $xyzt$,
x^2y, xyz^2</td></tr>
<tr><td>$W_{4,2}(\mathcal{V}) = \{x^3y, y^3x; x^2y^2\}$</td></tr>
<tr><td>$W_{4,3}(\mathcal{V}) = \{xyz^2, xzy^2, yzx^2\}$</td></tr>
<tr><td>(0.20)</td><td>$W_{3,2}(\mathcal{V}) = \{x^2y, y^2x; xyx\}$</td><td>$x$, x^2, x^3, xy, xyz</td></tr>
</table>

of length 3. Consequently, the group $\mathrm{Perm}_3(\mathcal{V})$ is distinct from a trivial one. It remains to take into account that every proper subgroup of $\mathbf{S}_3$ is a coatom in the lattice $\mathrm{Sub}(\mathbf{S}_3)$. By Lemma 2.8, for $n \geqslant 4$, $\mathrm{Perm}_n(\mathcal{V})$ contains one of the groups $\mathrm{Stab}_n(1)$ or $\mathrm{Stab}_n(n)$. It remains to take due account of the fact that all groups of the form $\mathrm{Stab}_n(i)$, where $1 \leqslant i \leqslant n$, are coatoms in $\mathrm{Sub}(\mathbf{S}_n)$.

We finish to prove the theorem. Let $\mathcal{V}$ be a semigroup variety specified by one of the systems (0.17)-(0.22). By Lemma 3.1 and Proposition 1.2, all 0-transversals like $W^0_{n,m}(\mathcal{V})$ are congruence-2.5-permutable. From our table we see that, for $1 < m < n$, at most one of non-empty transversals of the form $W_{n,m}(\mathcal{V})$ is not transitive. Since non-empty transversals like $W_{n,1}(\mathcal{V})$ and $W_{n,n}(\mathcal{V})$ are always transitive, $\mathcal{V}$ satisfies condition (b) of Lemma 1.11. Finally, using the data in the second column of Table 1 and keeping in mind that $xyzt = 0$ occurs in systems (0.17), (0.18), and (0.20) while $x_1x_2x_3x_4x_5 = 0$ occurs in (0.19), (0.21), and (0.22), we may spell out all (up to similarity) words that are not equal to 0 in $\mathcal{V}$ and belong to transitive transversals such as $W_{n,m}(\mathcal{V})$; cf. 3d column of the table. This data immediately implies that $\mathcal{V}$ satisfies condition (c) of Lemma 1.11. In view of this lemma, $\mathcal{V}$ is fi-2.5-permutable. The theorem is proved.

4. COROLLARIES

A direct consequence of the above theorem and Theorem 1 in [8] is the following:

COROLLARY 4.1. A semigroup variety, which is not a nil-variety, is fi-permutable if and only if it is fi-2.5-permutable.

It is of interest to compare this corollary with another fact, following immediately from [8, proof of Thm. 1], worded thus: a semigroup nil-variety is fi-permutable iff it is fi-1.5-permutable.

In [11] it was proved that a nil-variety has a distributive lattice of subvarieties iff it satisfies one of the systems (0.3)-(0.16), where in (0.3)-(0.8), π is as in the formulation of the theorem. (For a simpler and shorter proof of this fact, see [9]). This result and the theorem proved above can be combined to yield

COROLLARY 4.2. If $\mathcal{V}$ is a semigroup nil-variety with a distributive lattice of subvarieties then $\mathcal{V}$ is fi-2.5-permutable.

Recall that a semigroup variety is said to be *combinatorial* if it contains no non-trivial groups.

COROLLARY 4.3. Suppose that $\mathcal{V}$ is an fi-2.5-permutable semigroup variety and that one of the following two conditions holds:

(a) $\mathcal{V}$ is not a completely simple variety;

(b) $\mathcal{V}$ is a combinatorial variety.

Then $L(\mathcal{V}) \in \mathbf{M}_3$.

Proof. (a) In view of the theorem and the fact that $L(\mathcal{SL})$ is two-element, it suffices to handle the varieties specified by systems (0.3)-(0.22). As noted elsewhere above, in [11] it was proved that if a variety is specified by one of (0.3)-(0.16) then the lattice of its subvarieties is distributive. If, however, $\mathcal{V}$ is specified by one of (0.17)-(0.22) then we need only appeal to Propositions 1.1, 1.3 and Lemma 3.1.

(b) Addressing our theorem we see that every combinatorial fi-2.5-permutable variety either is a band variety or is a nil-variety. In the former case it suffices to take into account the fact that the lattice of all band varieties is distributive (see, e.g., [18]). In the latter case we need only look above at (a).

In connection with Corollary 4.3 it is worth mentioning yet another fact, following immediately from [8, Thm. 1], worded thus: if a semigroup variety $\mathcal{V}$ is fi-permutable, and one of the conditions in Corollary 4.3 holds, then $L(\mathcal{V})$ is a distributive lattice. It is also of interest to compare Corollary 4.3 with a result in [7] saying the following: if $\mathcal{V}$ is an overcommutative semigroup variety then subcommutative fully invariant congruences on $\mathcal{V}$-free semigroups are 2.5-permutable iff the lattice of overcommutative subvarieties of $\mathcal{V}$ belongs to $\mathbf{M}_3$.

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REFERENCES

1. G. Grätzer, *General Lattice Theory*, Academie-Verlag, Berlin (1978).
2. P. Lipparini, "n-Permutable varieties satisfy non trivial congruence identities," *Alg. Univ.*, **33**, No. 2, 159-168 (1995).
3. E. J. Tully, "The equivalence, for semigroup varieties, of two properties concerning congruence relations," *Bull. Am. Math. Soc.*, **70**, No. 3, 399-400 (1964).
4. F. J. Pastijn, "Commuting fully invariant congruences on free completely regular semigroups," *Trans. Am. Math. Soc.*, **323**, No. 1, 79-92 (1991).
5. M. Petrich and N. R. Reilly, "The modularity of the lattice of varieties of completely regular semigroups and related representations," *Glasg. Math. J.*, **32**, No. 2, 137-152 (1990).
6. M. V. Volkov and T. A. Ershova, "The lattice of varieties of semigroups with completely regular square," in *Semigroup Theory, Proc. Conf. Monash Univ.*, World Scientific, Singapore (1991), pp. 306-322.
7. B. M. Vernikov, "Distributivity, modularity, and related conditions in lattices of overcommutative semigroup varieties," in *Semigroups with Applications, Including Semigroup Rings, Proc. Int. Conf.*, Severny Ochag, St.-Petersburg (1999), pp. 411-439.
8. B. M. Vernikov and M. V. Volkov, "Permutability of fully invariant congruences on relatively free semigroups," *Acta Sc. Math.*, **63**, Nos. 3/4, 437-461 (1997).
9. B. M. Vernikov and M. V. Volkov, "Commuting fully invariant congruences on free semigroups," in *Contr. Gen. Algebra*, Vol. 12, Verlag Johannes Heyn, Klagenfurt (2000), pp. 391-417.

10. B. M. Vernikov, “Semigroup varieties with multiplicative restrictions on fully invariant congruences of their free objects,” *Dokl. Ross. Akad. Nauk*, **384**, No. 4, 446-448 (2002).

11. M. V. Volkov, “Semigroup varieties with commuting fully invariant congruences on free objects,” in *Algebra, Proc. Int. Conf., Cont. Math.*, **131**, Part 3, Am. Math. Soc., Providence, RI (1992), pp. 295-316.

12. B. M. Vernikov and M. V. Volkov, “Lattices of nilpotent semigroup varieties,” *Izv. Ur. Gos. Univ., Mat., Mekh.*, No. 10(1), 13-33 (1998).

13. B. M. Vernikov and M. V. Volkov, “The structure of lattices of nil-semigroup varieties,” *Izv. Ur. Gos. Univ., Mat., Mekh.*, No. 18(3), 34-52 (2000).

14. B. M. Vernikov, “On congruences of G-sets,” *Comm. Math. Univ. Carol.*, **38**, No. 3, 603-613 (1997).

15. R. N. McKenzie, G. F. McNulty, and W. F. Taylor, *Algebras. Lattices. Varieties*, Vol. 1, Wadsworth and Brooks/Cole, Monterey (1987).

16. E. Nelson, “The lattice of equational classes of semigroups with zero,” *Can. Math. Bull.*, **14**, No. 4, 531-534 (1971).

17. M. V. Sapir and E. V. Sukhanov, “Varieties of periodic semigroups,” *Izv. Vyssh. Uch. Zav., Ser. Mat.*, No. 4(227), 48-55 (1981).

18. T. Evans, “The lattice of semigroup varieties,” *Semigroup Forum*, **2**, No. 1, 1-43 (1971).

19. G. Pollák, “On the consequences of permutation identities,” *Acta Sc. Math.*, **34**, 323-333 (1973).