Semigroup varieties with 1.5-permutable fully invariant congruences on their free objects*

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Abstract

We describe semigroup varieties on whose free objects the product of any two fully invariant congruences coincides with their set-theoretical union.

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It is well known that a fulfillment of identities in lattices of varieties of universal algebras is closely related with multiplicative properties of fully invariant congruences on free algebras in varieties. (We take in mind properties which are formulated in terms of the multiplication of binary relations.) Let α and β be equivalence relations on the same set and n a positive integer. Put $\alpha \circ_n \beta = \alpha\beta\alpha\beta\cdots$ with n letters in the right side of the equality. Relations α and β are called *n-permutable* if $\alpha \circ_n \beta = \beta \circ_n \alpha$. Whenever $n = 2$ [respectively $n = 3$] we have a usual [*weak*] *permutability*. According to classical results due to B. Jonsson (see [3], for instance), if a variety $\mathcal{V}$ is [*weakly*] *congruence permutable* (that is, if any two congruences on every algebra of $\mathcal{V}$ are [weakly] permutable) then the subvariety lattice of $\mathcal{V}$ (which is denoted by $L(\mathcal{V})$) is arguesian [modular]. It is verified in [5, 6] that if a variety $\mathcal{V}$ is *congruence n-permutable* (that is, if any two congruences on every algebra of $\mathcal{V}$ are n-permutable) for some n then $L(\mathcal{V})$ satisfies a non-trivial lattice identity.

However, for the semigroup varieties, multiplicative restrictions imposed on all congruences of all semigroups of a variety turn out to be, as a rule, very rigid and are not of interest for the semigroup theory. So, in particular, a semigroup variety is congruence n-permutable if and only if it is a periodic group variety (it was verified in [11] for $n = 2$ and in [6] in the general case).

The situation becomes essentially more interesting if to impose the multiplicative restrictions not on all congruences but on the fully invariant ones and

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not on all semigroups of a variety but on the free objects of a variety only. In this case all the relationships with identities of varietal lattices are preserved completely, and, on the other hand, wide and important classes of varieties appear. So, for example, in [7] and [8] was proved independently that any two fully invariant congruences on every completely simple semigroups permute. In the same articles and in some other ones (see [20], for instance) results about multiplicative properties of fully invariant congruences on relatively free semigroups are successfully applied for examination of identities in lattices of varieties. In particular, in [7,8], it was proved just in such a way that the lattice of all completely regular semigroup varieties is modular and, moreover, arguesian.

It is well known that if α and β are congruences on the same algebra A then

$$\alpha \vee \beta = \alpha \cup \beta \cup \alpha\beta \cup \beta\alpha \cup \alpha\beta\alpha \cup \beta\alpha\beta \cup \cdots \cup \alpha \circ_n \beta \cup \beta \circ_n \alpha \cup \cdots , \qquad (1)$$

where $\vee$ is the join in the congruence lattice of A while $\cup$ is the set-theoretical union. It is clear that congruences α and β n-permute if and only if $\alpha \vee \beta = \alpha \circ_n \beta$. The equality (1) shows that it is natural to consider the following restriction on α and β:

$$\alpha \vee \beta = \alpha \circ_n \beta \cup \beta \circ_n \alpha \qquad (2)$$

(or, equivalently, $\alpha \circ_{n+1} \beta = \alpha \circ_n \beta \cup \beta \circ_n \alpha$). Clearly, this property is weaker than n-permutability but stronger than $(n+1)$-permutability. The equality (1) shows that (2) is, probably, a unique natural restriction on α and β that seats in the middle of n-permutability and $(n+1)$-permutability. Taking this in mind, we will call congruences α and β with the property (2) $n.5$-*permutable*. In particular, congruences α and β are 1.5-permutable if $\alpha \vee \beta = \alpha \cup \beta$ (equivalently, $\alpha\beta = \alpha \cup \beta$), and 2.5-permutable if $\alpha \vee \beta = \alpha\beta \cup \beta\alpha$ (equivalently, $\alpha\beta\alpha = \alpha\beta \cup \beta\alpha$). Put $\overline{\mathbb{N}} = \mathbb{N} \cup \{n + 0.5 \mid n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of all positive integers. Let $r \in \overline{\mathbb{N}}$. For brevity, we call a semigroup variety $\mathcal{V}$ fi-r-*permutable* if any two fully invariant congruences on every $\mathcal{V}$-free object r-permute; fi-r-permutable varieties with $r = 2$ [respectively $r = 3$] will be called [*weakly*] fi-*permutable*.

In [17] fi-permutable semigroup varieties were completely determined. A description of fi-2.5-permutable semigroup varieties is announced in [14] and will be published in [16]. A description of weakly fi-permutable varieties in some wide and important partial cases was announced in [14, 15]. Some other results related with the questions discussed here see in [12, 13, 18].

In this article we describe fi-1.5-permutable semigroup varieties. One can note that this restriction is also closely related with identities in subvariety lattices. Namely, it is very easy to check that any fi-1.5-permutable variety has a distributive subvariety lattice (see the proof of Corollary 3 below).

Recall that a semigroup variety $\mathcal{V}$ is called *chain* if the lattice $L(\mathcal{V})$ is a chain. Let $\mathcal{SL}$ be the variety of all semilattices and $\mathcal{LZ}$ [respectively $\mathcal{RZ}$] the variety of all left [right] zero semigroups. The main result of the article is the following theorem.

Theorem. *A semigroup variety $\mathcal{V}$ is fi-1.5-permutable if and only if either $\mathcal{V}$ is a chain variety of periodic groups or $\mathcal{V}$ coincides with one of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$ and $\mathcal{SL}$ or $\mathcal{V}$ satisfies one of the following identity systems:*

$$xyz = 0; \tag{3}$$
$$xyz = yxz,\ x^2 = 0; \tag{4}$$
$$xyz = xzy,\ x^2 = 0; \tag{5}$$
$$xyz = yzx,\ x^2 = 0; \tag{6}$$
$$xyz = zyx,\ x^2 = 0,\ xyx = 0; \tag{7}$$
$$xyz = zyx,\ x^2 = 0,\ xyzt = 0. \tag{8}$$

One can note that locally finite chain varieties of groups were completely determined in [1]. Thus, in the locally finite case, our Theorem gives the exhaust decription of fi-1.5-permutable semigroup varieties. On the other hand, the problem of complete description of arbitrary chain varieties of periodic groups seems to be of extraordinary difficult: results of [4] implies that, for any prime $p > 10^{216}$, there are uncountable many not locally finite group varieties of exponent p whose subvariety lattice is the 3-element chain.

Let us start with the proof of Theorem.

Necessity. The following general remark can be straightforwardly checked.

Lemma 1 *Let α, β, and ν be equivalences on a set S such that $\alpha, \beta \supseteq \nu$. Then α and β permute if and only if the equivalences α/ν and β/ν on the quotient set S/ν do so.* ■

By F we denote the free semigroup over a countable alphabet $\{x_1, x_2, \ldots\}$. Recall that a semigroup variety $\mathcal{V}$ is called *precomplete* if the lattice $L(\mathcal{V})$ has exactly one atom.

Lemma 2 *If a semigroup variety $\mathcal{V}$ is fi-1.5-permutable then $\mathcal{V}$ is precomplete.*

Proof. Suppose that the lattice $L(\mathcal{V})$ has two different atoms $\mathcal{A}$ and $\mathcal{B}$. Let α and β denote the fully invariant congruences on the semigroup F corresponding to the varieties $\mathcal{A}$ and $\mathcal{B}$ respectively. By Lemma 1 the congruences α and β 1.5-permute. Since the variety $\mathcal{A} \wedge \mathcal{B}$ is trivial, $\alpha \vee \beta$ is the universal relation on F. Let x and y are different letters of F. Then $(x, y) \in \alpha \vee \beta = \alpha \cup \beta$. This means that either $x\,\alpha\,y$ or $x\,\beta\,y$. In other words, one of the varieties $\mathcal{A}$ and $\mathcal{B}$ satisfies the identity $x = y$, contradicting the choice of these varieties. ■

It is well known that every precomplete semigroup variety is either a periodic group variety or one of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$ and $\mathcal{SL}$ or a nilsemigroup variety (see [9], for instance). Clearly, any fi-1.5-permutable variety is fi-permutable.

According to [17, Theorem 1], any fi-permutable nilsemigroup variety satisfies one of the identity systems (3)–(8). Lemma 2 shows that, to complete the proof of necessity, it remains to verify the following lemma.

Lemma 3 *If a periodic group variety $\mathcal{V}$ is fi-1.5-permutable then $\mathcal{V}$ is chain.*

Proof. Arguing by contradiction, suppose that $\mathcal{V}$ contains varieties $\mathcal{X}$ and $\mathcal{Y}$ that are non-comparable in the lattice $L(\mathcal{V})$. Let χ and η denote the fully invariant congruences on the semigroup F corresponding to the varieties $\mathcal{X}$ and $\mathcal{Y}$ respectively. By Lemma 1 the congruences χ and η are 1.5-permutable.

Since $\mathcal{X}$ and $\mathcal{Y}$ are non-comparable in $L(\mathcal{V})$, there are words u_1, v_1, u_2 and v_2 such that $(u_1, v_1) \in \chi \setminus \eta$ and $(u_2, v_2) \in \eta \setminus \chi$. Then $u_1u_2 \, \chi \, v_1u_2 \, \eta \, v_1v_2$, that is $(u_1u_2, v_1v_2) \in \chi\eta = \chi \cup \eta$. Suppose that $u_1u_2 \, \chi \, v_1v_2$. Then $u_1u_2 \, \chi \, v_1v_2 \, \chi \, u_1v_2$, that is $\mathcal{X}$ satisfies the identity $u_1u_2 = u_1v_2$. Since $\mathcal{X}$ is a group variety, we have that it satifies the identity $u_2 = v_2$. But this contradicts the choice of the words u_2 and v_2. It is verified quite analogously that if $u_1u_2 \, \eta \, v_1v_2$ then the identity $u_1 = v_1$ holds in $\mathcal{Y}$, contradicting with the choice of the words u_1 and v_1. ■

Sufficiency. It follows from the proof of [17, Proposition 1.15] that a nilsemigroup variety satisfying one of the identity systems (3)–(8) is fi-1.5-permutable. Since the varieties $\mathcal{LZ}$, $\mathcal{RZ}$ and $\mathcal{SL}$ are minimal non-trivial semigroup varieties (see [2], for instance), it remains to verify the following lemma.

Lemma 4 *If $\mathcal{V}$ is a chain semigroup variety then $\mathcal{V}$ is fi-1.5-permutable.*

Proof. Clearly, the lattice of fully invariant congruences on any $\mathcal{V}$-free semigroup is a chain, and therefore, any two fully invariant congruences on every such a semigroup 1.5-permute. ■

Theorem is proved. ■

As an immediate consequence of the Theorem we have the following corollary.

Corollary 1 *If a semigroup variety is not a nilsemigroup one then it is fi-1.5-permutable if and only if it is chain.* ■

An analogue of Corollary 1 for nilsemigroup varieties is not valid. Indeed, the description of non-group chain varieties of semigroups given in [10] shows that the variety given by any of the identity systems (3)–(8) is not chain (this may be easily verified also by direct calculations without references to [10]).

Theorem proved above and Theorem 1 of [17] imply the following corollary.

Corollary 2 *If a semigroup variety is not a completely simple one then it is fi-permutable if and only if it is fi-1.5-permutable.* ■

In particular, fi-permutability and fi-1.5-permutability are equivalent for nilsemigroup varieties. It is interesting to compare this fact with the following one: without of nil-case fi-permutability is equivalent to fi-2.5-permutability [16, Corollary 4.1].

In the article [19] nilsemigroup varieties with distributive subvariety lattice was completely determined (a shorter and simpler proof of this result see in [18]). This result together with [17, Theorem 1] imply that any fi-permutable nilsemigroup variety has a distributive subvariety lattice. Corollary 2 permits to give a short and simple proof of this fact without using of results of the article [19].

Corollary 3 *If a semigroup variety is not a completely simple one and is fi-permutable then its subvariety lattice is distributive.*

Proof. Let $\mathcal{V}$ be an fi-permutable but not completely simple semigroup variety, S the $\mathcal{V}$-free object of a countable rank and L the lattice of fully invariant congruences on S. By Corollary 2 $\mathcal{V}$ is fi-1.5-permutable. Hence L is a sublattice in the subset lattice of $S \times S$, and therefore L is distributive. Since the lattices $L(\mathcal{V})$ and L are antiisomorphic, the former lattice is distributive too. ∎

One can note that without completely regular case some stronger version of Corollary 3 holds. Namely, if a semigroup variety $\mathcal{V}$ is not completely regular then the lattice $L(\mathcal{V})$ is distributive whenever, on every $\mathcal{V}$-free object S, any two fully invariant congruences contained in the least semilattice congruence on S permute [18, Corollary 4.1].

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